1 The Basic Representation

We define the imaginary quantity $i$ to be

\[ i = \sqrt{-1}. \tag{1.1} \]

Any complex number can be written in the standard form

\[ z = a + ib, \tag{1.2} \]

where $a$ and $b$ are real numbers. The real part of $z$, denoted Re $z$, is the real number $a$. The imaginary part of $z$, denoted Im $z$, is the real number $b$. The complex conjugate of $z$, written $z^*$ (or sometimes $\bar{z}$), is the complex number...
obtained from $z$ by flipping the sign of the imaginary part:

$$z = a + ib \Rightarrow z^* = a - ib. \quad (1.3)$$

Complex numbers are added, subtracted, multiplied, and divided just like real numbers. So

$$(a + ib) + (c + id) = (a + c) + i(b + d), \quad (1.4)$$

and

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc). \quad (1.5)$$

The ratio of two complex numbers can be put into standard form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\frac{a + ib}{c + id} = \frac{a + ib}{c + id} \frac{(c - id)}{(c - id)} = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2} = \left( \frac{ac + bd}{c^2 + d^2} \right) + i \left( \frac{ad - bc}{c^2 + d^2} \right). \quad (1.6)$$

2 The Modulus

Note that the product of a complex number and its complex conjugate is always a real number:

$$z = a + ib \Rightarrow zz^* = a^2 + b^2. \quad (2.1)$$
The modulus of $z$, written $|z|$, is defined to be the positive root of $zz^*$:

$$|z| = (zz^*)^{1/2}. \quad (2.2)$$

A short calculation shows that the modulus is multiplicative, meaning that, for any two complex numbers $z_1$ and $z_2$,

$$|z_1 z_2| = |z_1||z_2|. \quad (2.3)$$

Similarly,

$$\frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|}. \quad (2.4)$$

3 A Geometric Interpretation

There is a one-to-one correspondence between complex numbers and points of the plane, given by

$$a + ib \leftrightarrow (a, b), \quad (3.1)$$

and illustrated in Figure 1.
This allows us to specify a complex number by its polar coordinates, usually denoted \((r, \theta)\). From elementary trigonometry it is clear that

\[
a = r \cos \theta \quad r = (a^2 + b^2)^{1/2} = |z| \tag{3.2}
\]

\[
b = r \sin \theta \quad \theta = \tan^{-1} \left( \frac{b}{a} \right). \tag{3.3}
\]

\(r\) (which is just the modulus of \(z\)) is sometimes called the length of \(z\), and \(\theta\) is called the argument (or angle) of \(z\), and is denoted \(\text{Arg } z\).

### 4 The Complex Exponential Representation

Complex numbers admit another representation in terms of complex exponentials. To motivate this, recall that, near \(x = 0\), a differentiable function can be approximated by its MacLaurin-Taylor expansion

\[
f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots. \tag{4.1}
\]

Applying this formula enables us to write

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{4.2}
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{4.3}
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots. \tag{4.4}
\]
Substituting \( x = i\theta \) in the expansion of \( e^x \) and collecting real and imaginary parts gives

\[
e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots
\]

\[
= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right),
\]

whereupon we obtain Euler’s formula:

\[
e^{i\theta} = \cos \theta + i\sin \theta. \tag{4.5}
\]

Among other things, Euler’s formula yields the classic equality

\[e^{i\pi} = -1.\]

Using (3.2) and (3.3) we can write any complex number as

\[z = a + ib = r \cos \theta + ir \sin \theta, \tag{4.6}\]

so comparing with (4.5) we see that any complex number can be written in complex exponential form

\[z = re^{i\theta}. \tag{4.7}\]

As a consequence of Euler’s formula we get the very useful equations:

\[
\cos \theta = \frac{1}{2} \left(e^{i\theta} + e^{-i\theta}\right) \tag{4.8}
\]

\[
\sin \theta = \frac{1}{2i} \left(e^{i\theta} - e^{-i\theta}\right). \tag{4.9}
\]

The complex exponential representation is ideally suited to multiplication
and division of complex numbers because of the corresponding properties of the exponential. In particular, for any two complex numbers

\[ z_1 = r_1 e^{i\theta_1} \quad \text{and} \quad z_2 = r_2 e^{i\theta_2}, \quad (4.10) \]

we have

\[ z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \quad (4.11) \]

Geometrically, the product of \( z_1 \) and \( z_2 \) has length equal to the product of the lengths of \( z_1 \) and \( z_2 \) and argument equal to the sum of the arguments of \( z_1 \) and \( z_2 \).

## 5 Roots of Unity

According to the Fundamental Theorem of Algebra, an \( n \)th order polynomial has \( n \) complex roots. This means that the equation

\[ z^n = 1 \quad (5.1) \]

has \( n \) solutions, which are called \( n \)th roots of unity. To find them, we use Euler’s formula (4.5) to observe that, for any integer \( k \),

\[ e^{2\pi ki} = 1. \quad (5.2) \]

Thus (5.1) can be solved by writing

\[ z = (1)^{1/n} = (e^{2\pi ki})^{1/n} = e^{2\pi(k/n)i}. \quad (5.3) \]
Figure 2: The three roots of unity

Observe that we get distinct solutions only for the values $k = 0, 1, 2, \ldots, n-1$, after which the solutions repeat. Equivalently, we get distinct solutions only for the integers modulo $n$. Hence, the $n^{th}$ roots of unity are

$$1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \ldots, e^{2(n-1)\pi i/n}. \quad (5.4)$$

**Example 1** The cube roots of unity are

$$1, e^{2\pi i/3}, e^{4\pi i/3}.$$  

Equivalently, they are

$$1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \text{ and } -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \quad (5.5)$$

(See Figure 2.) Note that they are evenly spaced around the unit circle.

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1Recall that, if $a$ and $b$ are integers and $n$ is a natural number, we say $a$ is congruent to $b$ modulo $n$, written $a \equiv b \mod n$, if $a - b$ is evenly divisible by $n$.  

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Example 2  The cube roots of $2i$ can be obtained by rewriting $2i$ in complex exponential representation:

$$(2i)^{1/3} = \left(2e^{i\pi/2 + 2\pi ki}\right)^{1/3} = 2^{1/3}e^{i\pi/6 + 2\pi(k/3)i}, \quad k = 0, 1, 2.$$ 

Equivalently, we may write

$$2^{1/3}e^{i\pi/6}, 2^{1/3}e^{5\pi i/6}, \text{ and } 2^{1/3}e^{9\pi i/6}.$$ 

These, too, are evenly spaced around a circle (although in this case the circle has radius $2^{1/3}$).

It would appear that the $n^{th}$ roots of any complex number are always spaced evenly around a circle. Equivalently, their sum always seems to be zero. In fact, a more general result holds.

**Theorem.** Let $\xi_j := e^{2\pi ij/n}, \ j = 0, 1, 2, \ldots, n-1$ be the $n^{th}$ roots of unity. Then

$$\frac{1}{n} \sum_{j=0}^{n-1} \xi_j^k = \delta_{k,0} = \begin{cases} 1, & \text{if } k = 0 \text{ mod } n, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof.** The case $k = 0$ is evident, so suppose $k \neq 0$. Recall the expression for the sum of the finite geometric series

$$\sum_{j=0}^{N-1} x^j = \frac{1 - x^N}{1 - x},$$

which can easily be proved by multiplying both sides by $(1 - x)$. Then we
have
\[
\sum_{j=0}^{n-1} e^k_j = \sum_{j=0}^{n-1} e^{2\pi i jk/n} = \sum_{j=0}^{n-1} \left( e^{2\pi ik/n} \right)^j = \frac{1 - e^{2\pi ik}}{1 - e^{2\pi ik/n}} = 0,
\]
because when \( k \neq 0 \mod n \) the numerator vanishes but denominator does not. \( \square \)