Complex Numbers

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1 The Basic Representation

We define the imaginary quantity i to be

$$i = \sqrt{-1}. \tag{1.1}$$

Any complex number can be written in the *standard form*

$$z = a + ib, \tag{1.2}$$

where a and b are real numbers. The *real part* of z, denoted $\operatorname{Re} z$, is the *real* number a. The *imaginary part* of z, denoted $\operatorname{Im} z$, is the *real* number b. The *complex conjugate* of z, written z^* (or sometimes $\bar{z}$), is the complex number

obtained from z by flipping the sign of the imaginary part:

$$z = a + ib \quad \Rightarrow \quad z^* = a - ib. \tag{1.3}$$

Complex numbers are added, subtracted, multiplied, and divided just like real numbers. So

$$(a + ib) + (c + id) = (a + c) + i(b + d), \tag{1.4}$$

and

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc). \tag{1.5}$$

The ratio of two complex numbers can be put into standard form by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$\begin{aligned} \frac{a + ib}{c + id} &= \frac{a + ib}{c + id}\left(\frac{c - id}{c - id}\right) \\ &= \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2} \\ &= \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{ad - bc}{c^2 + d^2}\right). \end{aligned} \tag{1.6}$$

2 The Modulus

Note that the product of a complex number and its complex conjugate is always a real number:

$$z = a + ib \quad \Rightarrow \quad zz^* = a^2 + b^2. \tag{2.1}$$

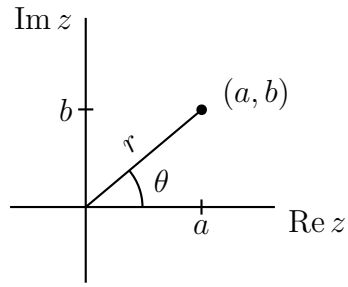

Figure 1: The complex plane

The *modulus* of z, written $|z|$, is defined to be the *positive root* of zz^*:

$$|z| = (zz^*)^{1/2}. \tag{2.2}$$

A short calculation shows that the modulus is multiplicative, meaning that, for any two complex numbers z_1 and z_2,

$$|z_1 z_2| = |z_1||z_2|. \tag{2.3}$$

Similarly,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}. \tag{2.4}$$

3 A Geometric Interpretation

There is a one-to-one correspondence between complex numbers and points of the plane, given by

$$a + ib \qquad \longleftrightarrow \qquad (a, b), \tag{3.1}$$

and illustrated in Figure 1.

This allows us to specify a complex number by its polar coordinates, usually denoted (r, θ). From elementary trigonometry it is clear that

$$a = r\cos\theta \qquad r = (a^2 + b^2)^{1/2} = |z| \tag{3.2}$$

$$b = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{b}{a}\right). \tag{3.3}$$

r (which is just the modulus of z) is sometimes called the *length* of z, and θ is called the *argument* (or *angle*) of z, and is denoted $\operatorname{Arg} z$.

4 The Complex Exponential Representation

Complex numbers admit another representation in terms of complex exponentials. To motivate this, recall that, near $x = 0$, a differentiable function can be approximated by its MacLaurin-Taylor expansion

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \cdots . \tag{4.1}$$

Applying this formula enables us to write

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{4.2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{4.3}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots . \tag{4.4}$$

Substituting $x = i\theta$ in the expansion of e^x and collecting real and imaginary parts gives

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right), \end{aligned}$$

whereupon we obtain *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{4.5}$$

Among other things, Euler's formula yields the classic equality

$$e^{i\pi} = -1.$$

Using (3.2) and (3.3) we can write any complex number as

$$z = a + ib = r\cos\theta + ir\sin\theta, \tag{4.6}$$

so comparing with (4.5) we see that any complex number can be written in *complex exponential form*

$$z = re^{i\theta}. \tag{4.7}$$

As a consequence of Euler's formula we get the very useful equations:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \tag{4.8}$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \tag{4.9}$$

The complex exponential representation is ideally suited to multiplication

and division of complex numbers because of the corresponding properties of the exponential. In particular, for any two complex numbers

$$z_1 = r_1 e^{i\theta_1} \qquad \text{and} \qquad z_2 = r_2 e^{i\theta_2}, \tag{4.10}$$

we have

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \qquad \text{and} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}. \tag{4.11}$$

Geometrically, the product of z_1 and z_2 has length equal to the product of the lengths of z_1 and z_2 and argument equal to the sum of the arguments of z_1 and z_2.

5 Roots of Unity

According to the Fundamental Theorem of Algebra, an nth order polynomial has n complex roots. This means that the equation

$$z^n = 1 \tag{5.1}$$

has n solutions, which are called n^{th} *roots of unity*. To find them, we use Euler's formula (4.5) to observe that, for any integer k,

$$e^{2\pi k i} = 1. \tag{5.2}$$

Thus (5.1) can be solved by writing

$$z = (1)^{1/n} = \left(e^{2\pi k i}\right)^{1/n} = e^{2\pi(k/n)i}. \tag{5.3}$$

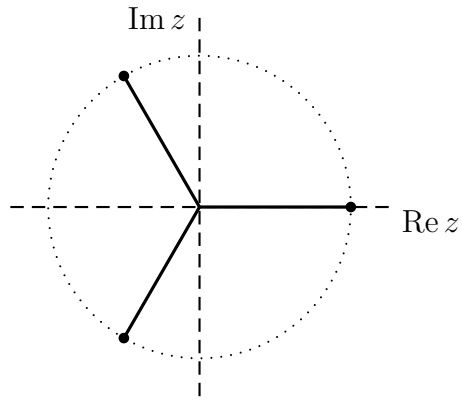

Figure 2: The three roots of unity

Observe that we get distinct solutions only for the values $k = 0, 1, 2, \ldots, n-1$, after which the solutions repeat. Equivalently, we get distinct solutions only for the integers modulo n. [1] Hence, the nth roots of unity are

$$1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \ldots, e^{2(n-1)\pi i/n}. \tag{5.4}$$

Example 1 The cube roots of unity are

$$1, e^{2\pi i/3}, \text{ and } e^{4\pi i/3}.$$

Equivalently, they are

$$1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \text{ and } -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \tag{5.5}$$

(See Figure 2.) Note that they are evenly spaced around the unit circle.

[1] Recall that, if a and b are integers and n is a natural number, we say a *is congruent to* b *modulo* n, written $a = b \mod n$, if $a - b$ is evenly divisible by n.

Example 2 The cube roots of $2i$ can be obtained by rewriting $2i$ in complex exponential representation:

$$(2i)^{1/3} = \left(2e^{i\pi/2+2\pi ki}\right)^{1/3} = 2^{1/3}e^{i\pi/6+2\pi(k/3)i}, \quad k = 0, 1, 2.$$

Equivalently, we may write

$$2^{1/3}e^{i\pi/6}, 2^{1/3}e^{5\pi i/6}, \text{ and } 2^{1/3}e^{9\pi i/6}.$$

These, too, are evenly spaced around a circle (although in this case the circle has radius $2^{1/3}$).

It would appear that the nth roots of any complex number are always spaced evenly around a circle. Equivalently, their sum always seems to be zero. In fact, a more general result holds.

Theorem. *Let $\xi_j := e^{2\pi ij/n}$, $j = 0, 1, 2, \ldots, n-1$ be the nth roots of unity. Then*

$$\frac{1}{n}\sum_{j=0}^{n-1} \xi_j^k = \delta_{k,0} = \begin{cases} 1, & \textit{if } k = 0 \mod n, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The case $k = 0$ is evident, so suppose $k \neq 0$. Recall the expression for the sum of the *finite geometric series*

$$\sum_{j=0}^{N-1} x^j = \frac{1 - x^N}{1 - x},$$

which can easily be proved by multiplying both sides by $(1 - x)$. Then we

have

$$\sum_{j=0}^{n-1} \xi_j^k = \sum_{j=0}^{n-1} e^{2\pi ijk/n} = \sum_{j=0}^{n-1} \left(e^{2\pi ik/n}\right)^j = \frac{1 - e^{2\pi ik}}{1 - e^{2\pi ik/n}} = 0,$$

because when $k \neq 0 \mod n$ the numerator vanishes but denominator does not. $\square$