Kinematics in the Plane

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A particle moving in the plane is described in Cartesian coordinates by a parameterized curve of the form [1]

$$\boldsymbol{r}(t) = (x(t), y(t)) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}}. \tag{1}$$

Because the Cartesian unit vectors do not change as the particle moves, the velocity of the particle can be written

$$\boldsymbol{v}(t) = (\dot{x}(t), \dot{y}(t)) = \dot{x}(t)\hat{\boldsymbol{\imath}} + \dot{y}(t)\hat{\boldsymbol{\jmath}}, \tag{2}$$

and the acceleration vector becomes

$$\boldsymbol{a}(t) = (\ddot{x}(t), \ddot{y}(t)) = \ddot{x}(t)\hat{\boldsymbol{\imath}} + \ddot{y}(t)\hat{\boldsymbol{\jmath}}. \tag{3}$$

[1] In what follows t is an arbitrary parameter, usually (but not always) denoting time.

These equations take a very different form when expressed in polar coordinates because in that case the unit vectors *do* change as the particle moves. In particular, let us write $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ for the unit vectors in the radial and tangential directions. To discover how these vectors change as the particle moves, we begin by expressing them in terms of the Cartesian unit vectors. Recall that

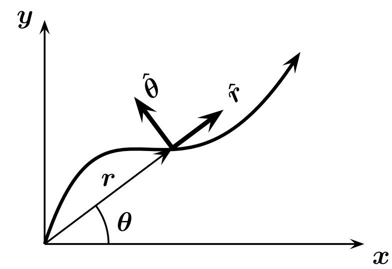

Figure 1: A Particle Trajectory

$$x = r\cos\theta, \tag{4}$$

$$y = r\sin\theta, \tag{5}$$

so we can write

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}. \tag{6}$$

Next comes the tricky part. We identify $\partial/\partial x$ with $\hat{\boldsymbol{\imath}}$, $\partial/\partial y$ with $\hat{\boldsymbol{\jmath}}$, and $\partial/\partial r$ with $\hat{\boldsymbol{r}}$ (up to scaling). This is because a vector field is really the same thing as a directional derivative. [2] In any event, (6) becomes

$$\boxed{\hat{\boldsymbol{r}} = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}}. \tag{7}$$

Similar reasoning yields

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}. \tag{8}$$

Using the same identifications as above, together with a new one identifying $\partial/\partial\theta$ with $\hat{\boldsymbol{\theta}}$ would give

$$\hat{\boldsymbol{\theta}} \stackrel{?}{=} -r\sin\theta\,\hat{\boldsymbol{\imath}} + r\cos\theta\,\hat{\boldsymbol{\jmath}}, \tag{9}$$

[2] It is easy enough to avoid this identification technique when finding $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$, but the method is worth knowing because it generalizes to more complicated situations.

which is incorrect. The reason is that $\hat{\boldsymbol{\theta}}$ is supposed to be a unit vector, whereas the right side has length r. This is easily remedied, however. Just divide the right side by r to get

$$\boxed{\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}}, \tag{10}$$

which is now correct.

It follows that from (7) that

$$\dot{\hat{\boldsymbol{r}}} = -\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\hat{\boldsymbol{\jmath}}, \tag{11}$$

or, comparing with (10),

$$\boxed{\dot{\hat{\boldsymbol{r}}} = \dot{\theta}\hat{\boldsymbol{\theta}}}. \tag{12}$$

Similarly, from (10) we get

$$\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\cos\theta\hat{\boldsymbol{\imath}} - \dot{\theta}\sin\theta\hat{\boldsymbol{\jmath}}. \tag{13}$$

Comparing with (7) gives

$$\boxed{\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\hat{\boldsymbol{r}}}. \tag{14}$$

Taking derivatives of (12) and (14) and using those very equations again gives

$$\boxed{\ddot{\hat{\boldsymbol{r}}} = -\dot{\theta}^2\hat{\boldsymbol{r}} + \ddot{\theta}\hat{\boldsymbol{\theta}}} \tag{15}$$

and

$$\boxed{\ddot{\hat{\boldsymbol{\theta}}} = -\ddot{\theta}\hat{\boldsymbol{r}} - \dot{\theta}^2\hat{\boldsymbol{\theta}}}. \tag{16}$$

At last we can compute the velocity and acceleration vectors in polar coordinates. The particle's position is described by the vector $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, so differentiating with respect to the parameter t gives

$$\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}. \tag{17}$$

Using (12) we find

$$\boxed{\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}}. \tag{18}$$

Notice that this makes perfect sense: the velocity component in the radial direction is just $\dot{r}$, while the velocity component in the tangential direction is $r\dot{\theta}$, which is r times the angular velocity. Differentiating (18) and using (12) and (14) yet again gives

$$\boldsymbol{a} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\hat{\boldsymbol{r}}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\theta}\dot{\hat{\boldsymbol{\theta}}} = \ddot{r}\hat{\boldsymbol{r}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{r}\dot{\theta}\hat{\boldsymbol{\theta}} + r\ddot{\theta}\hat{\boldsymbol{\theta}} - r\dot{\theta}^2\hat{\boldsymbol{r}}. \tag{19}$$

Collecting terms yields

$$\boxed{\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}}. \tag{20}$$

The radial component of the acceleration has two terms: the $\ddot{r}$ term comes from changes in the radial distance, while the $r\dot{\theta}^2$ is just the centripetal acceleration term. (Observe that it is directed radially inward, as required.) The tangential component of the acceleration also has two terms. The first term is just the usual tangential acceleration, while the second term is the coriolis acceleration.