On the spectra of abelian rook graphs

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Abstract

We define a new family of graphs, called *abelian rook graphs*. These are Cayley graphs on the abelian group $\mathbb{Z}_n^d$ with a specific connection set that mimics the adjacency condition of the simplicial rook graphs. We investigate some of their basic properties and determine their spectra completely. The spectra turn out to be integral.

Keywords: Cayley graph, simplicial rook graph, abelian rook graph, spectrum

1 Introduction

Let $C_{n,d}$ denote the set of weak d-compositions of n. That is,

$$C_{n,d} = \{(x_1, \dots, x_d) : \sum_{i=1}^{d} x_i = n \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq d\}.$$

We define a graph $S(n, d)$ on $C_{n,d}$ by joining two compositions if they differ in precisely two entries. The graph $S(n, d)$ is called a *simplicial rook graph*, because the vertices can be identified with the lattice points inside the n^{th} dilate of the standard simplex, where two points are joined by an edge if they lie upon the same lattice line. The terminology arises because we may view two points joined by an edge as a pair of rooks on a simplicial chessboard. In particular, the number of non-attacking rooks on such a chessboard is the independence number of the graph.

Simplicial rook graphs we first defined and studied by Martin and Wagner [12], although the independence number of $S(n,3)$, namely $\lfloor(2n+3)/3\rfloor$, was obtained earlier (and independently) by Blackburn, Paterson, and Stinson [1] and Nivasch and Lev [13]. It is not difficult to see ([12], Prop. 2.1) that $S(n,d)$ has $\binom{n+d-1}{d-1}$ vertices and is regular of degree $n(d-1)$. Martin and Wagner determined the spectrum of $S(n,3)$ for all n, and, on the basis of computational evidence, conjectured that the spectra of all the graphs $S(n,d)$ are integral. They also conjectured that the least eigenvalue of $S(n,d)$ is equal to $\max\{-n,-\binom{d}{2}\}$, and gave conjectured values for some of their multiplicities. [1] All of these conjectures were subsequently confirmed by Brouwer, Cioabă, Haemers, and Vermette [3], who also proved several other interesting results about simplicial rook graphs.

Many questions about these graphs remain unanswered, such as the general eigenvalue spectrum of $S(n,d)$ and the exact independence number of $S(n,d)$. These appear to be difficult problems, especially because, as was pointed out by Martin and Wagner, these graphs do not seem to have nice graph theoretical characterizations; in particular, they are generally neither vertex transitive nor distance regular.

In this work we introduce a related class of graphs inspired by the simplicial rook graphs, but possessing much more structure. These new graphs, which we call *abelian rook graphs*, have much nicer properties than the simplicial rook graphs, and accordingly afford a much simpler analysis. In particular, the abelian rook graphs are Cayley graphs, hence vertex transitive, and their entire spectrum can be determined. The spectrum turns out to be integral as well.

It should be emphasized from the outset that, other than the similarity of their origins, these two classes of graph are in fact quite different. In particular, their spectra are not at all similar; they do not even have the same number of vertices. We study them here

[1] Least eigenvalues are of interest in their own right, but also because they are connected to the independence number α of a graph *via* the Hoffman bound, which states that $\alpha \leq |V|/(1-k/\tau)$, where $|V|$ is the number of vertices, k is the degree, and τ is the least eigenvalue. But the Hoffman bound is not always exact. For instance, for $S(n,3)$ with $n>3$, the Hoffman bound gives $\alpha < 3(n+2)(n+1)/(4n+6)$, which is weaker than the actual bound given above.

for their intrinsic interest, but it is hoped that the reformulation of simplicial rook graphs given in the next section, which serves as the motivation for abelian rook graphs, may be of some utility in addressing questions about simplicial rook graphs.

2 Raising and lowering operators

To motivate the introduction of the abelian rook graphs, we first reformulate the definition of the simplicial rook graphs in terms of raising and lowering operators acting on tensor product spaces. Let V be an $(n+1)$-dimensional real inner product space. We will use Dirac notation, so that elements of V are denoted by ket vectors $|x\rangle$, and elements of the dual space are denoted by bra vectors $\langle x|$. The inner product of $|x\rangle$ and $|y\rangle$ is denoted $\langle x|\, y\rangle$. The canonical orthonormal basis of V consists of the vectors $\{|0\rangle, |1\rangle, \ldots, |n\rangle\}$.

Define two linear operators, L and R, acting on the basis elements by

$$L\,|0\rangle = 0 \quad \text{and} \quad L\,|x\rangle = |x-1\rangle\,, \quad (1 \leq x \leq n)$$

and

$$R\,|n\rangle = 0 \quad \text{and} \quad R\,|x\rangle = |x+1\rangle\,, \quad (0 \leq a \leq x-1).$$

We call L and R, *lowering* and *raising* operators, respectively. It is easy to see that they are adjoints of one another with respect to the inner product. That is $L = R^\dagger$, where by definition, $\left\langle Z^\dagger x\right|\, y\rangle = \langle x|\, Zy\rangle$. Observe that

$$[L,R]\,|0\rangle = |0\rangle \quad \text{and} \quad [L,R]\,|n\rangle = -\,|n\rangle \quad \text{and} \quad [L,R]\,|x\rangle = 0 \qquad 1 \leq x \leq n-1, \quad (1)$$

where $[L,R] := LR - RL$ is the usual commutator. In particular, $[L,R] \neq 0$.

Let $\mathcal{H}$ denote the d-fold tensor product space $V^{\otimes d}$, having dimension $(n+1)^d$. Elements of $\mathcal{H}$ consist of linear combinations of vectors of the form

$$|x_1, \ldots, x_d\rangle := |x_1\rangle \otimes \cdots \otimes |x_d\rangle\,.$$

The space $\mathcal{H}$ inherits the natural inner product

$$\langle x_1, \ldots, x_d|\, y_1, \ldots, y_d\rangle = \langle x_1|\, y_1\rangle \cdots \langle x_d|\, y_d\rangle\,.$$

Define the *sum operator* $S : \mathcal{H} \to \mathcal{H}$ by

$$S\,|x_1, \ldots, x_d\rangle = \left(\sum_{i=1}^{d} x_i\right) |x_1, \ldots, x_d\rangle\,,$$

extended by linearity. This map is Hermitian, so by the spectral theorem we can write

$$\mathcal{H} = \bigoplus_{c=0}^{nd} U_c, \tag{2}$$

where U_c is the subspace on which S takes the constant value c. By simple counting, we have $\dim U_c = \binom{c+d-1}{c}$.

For any operator Z on V, we use the shorthand notation

$$Z_i := 1 \otimes \cdots \otimes \underbrace{Z}_{i} \otimes \cdots \otimes 1,$$

to denote the corresponding action on the i^{th} component of the tensor product space $\mathcal{H}$. Clearly, $[Z_i, Z_j] = 0$ if $i \neq j$. Next, we construct the following operator on $\mathcal{H}$:

$$H := \sum_{1 \leq i < j \leq d} \sum_{k=1}^{n} (L_i^k R_j^k + R_i^k L_j^k) = \sum_{1 \leq i \neq j \leq d} \sum_{k=1}^{n} L_i^k R_j^k. \tag{3}$$

Explicitly, we have

$$H \left| x_1, \ldots, x_d \right\rangle = \sum_{1 \leq i \neq j \leq d} \sum_{k=1}^{n} \left| x_1, \ldots, x_i - k, \ldots, x_j + k, \ldots, x_d \right\rangle .$$

Evidently, $[H, S] = 0$, so H respects the subspace decomposition (2). A moment's thought reveals that $H_n := H|_{U_n}$ is precisely the adjacency operator of the simplicial rook graph $S(n, d)$, where we identify the weak d-compositions of n with the kets $|x_1, \ldots, x_d\rangle$ lying in the subspace U_n. [2]

3 Abelian rook graphs

The eigenanalysis of H_n is complicated by the fact that $[L_i, R_i] \neq 0$ for any i. For this reason, we are motivated to change the ground rules and introduce a new class of graphs,

[2] The adjacency operator $\hat{A}$ of a graph on a vertex set X is just the linear operator on the vector space $\mathbb{R}^X$ of functions $f : X \to \mathbb{R}$, defined by $(\hat{A}f)(x) = \sum_{y \sim x} f(y)$, where $y \sim x$ means y and x are adjacent. The matrix representation of $\hat{A}$ in the basis of characteristic functions on the vertices is the adjacency matrix A of the graph. We may therefore speak of the spectrum of either the adjacency matrix or the adjacency operator, as suits us.

which we denote by $\Gamma(n,d)$. (These are not yet the graphs we want.) The idea is to impose *periodic boundary conditions* on the kets, so that $|x\rangle$ is identified with $|x+n\rangle$. [3] The space V is now n-dimensional, spanned by vectors of the form $|0\rangle, |1\rangle, \ldots, |n-1\rangle$. The lowering and raising operators are now given by

$$L|x\rangle = |x-1\rangle \quad \text{and} \quad R|x\rangle = |x+1\rangle \qquad (0 \le x \le n-1),$$

whereupon we see that $[L,R]=0$. As before, we define $\mathcal{H} := V^{\otimes d}$, so that $\mathcal{H}$ is essentially a discrete torus. The sum operator S is defined exactly as before, except that now everything is computed modulo n. Again, we have $\mathcal{H} = \bigoplus_c U_c$, where U_c is the subspace of $\mathcal{H}$ on which S takes the value $c \bmod n$.

The vertices of $\Gamma(n,d)$ are identified with kets of the form $|x_1,\ldots,x_d\rangle$, where now $0 \le x_i \le n-1$ for $1 \le i \le d$. Two vertices of $\Gamma(n,d)$ are adjacent if they differ by a vector of the form $|0,\ldots,k,\ldots,-k,\ldots,0\rangle$ for $1 \le k \le n-1$. The *abelian rook graph* $T(n,d)$ is the subgraph of $\Gamma(n,d)$ induced by the vertices in U_0. The adjacency operator of $\Gamma(n,d)$ on $\mathcal{H}$ is

$$A := \sum_{1\le i<j\le d} \sum_{k=1}^{n} L_i^k R_j^k, \tag{4}$$

so the adjacency operator of $T(n,d)$ is $A_n := A|_{U_0}$. Because L_i and R_i now commute, the spectrum of the adjacency operator A simplifies considerably.

4 Abelian rook graphs as Cayley graphs on $\mathbb{Z}_n^d$

Having motivated their introduction, we may now describe abelian rook graphs in simpler terms. Specifically, we may view them as Cayley graphs on an abelian group (whence the nomenclature). Let G be a group, and suppose $C \subseteq G$ is inverse closed (so that $g \in C \Rightarrow g^{-1} \in C$) and that $1 \notin C$. The Cayley graph $X(G,C)$ is the graph whose vertices are the elements of G, with $g \sim h$ if $gh^{-1} = c$ for some $c \in C$. [4] The set C is the *connection set* of $X(G,C)$. The graph $\Gamma(n,d)$ may be viewed as a Cayley graph on the

[3] In the interest of notational brevity, and at the risk of confusing the reader, we want to identify $|x\rangle = |x+n\rangle$ rather than $|x+n+1\rangle$. This makes the notation simpler, but means that, if we were to compare simplicial rook graphs and abelian rook graphs, we might need to shift n by unity, depending upon what is to be compared.

[4] The conditions on C ensure that $X(G,C)$ is undirected and without loops.

abelian group

$$G := \underbrace{\mathbb{Z}_n \times \mathbb{Z}_n \times \cdots \mathbb{Z}_n}_{d \text{ times}},$$

elements of which are of the form $(x_1, \ldots, x_d)$ with $x_i \in \mathbb{Z}_n$, with connection set [5]

$$C := \bigcup_{\substack{1 \le i < j \le d \\ 0 \le k \le n-1}} (0, \ldots, \underbrace{+k}_{i}, \ldots, \underbrace{-k}_{j}, \ldots, 0).$$

We take a moment to discuss some properties of $\Gamma(n, d)$ and $T(n, d)$.

Proposition 1. *$\Gamma(n, d)$ has n^d vertices and n connected components, each of which is isomorphic to $T(n, d)$ (which therefore has n^{d-1} vertices). The graph $T(n, d)$ has diameter $d - 1$.*

Proof. The graph $\Gamma(n, d)$ clearly has n^d vertices. For any vertex $(x_1, \ldots, x_d) \in \Gamma(n, d)$ we call $\sum_i x_i \bmod n$ its *value*. Let $\Gamma_j(n, d)$ be the induced subgraph of $\Gamma(n, d)$ on vertices of value $j \bmod n$. In particular, $\Gamma_0(n, d) = T(n, d)$. There are no edges between any of these induced subgraphs, because the adjacency relations preserve value, so there are at least n connected components. We may choose a vertex of $\Gamma_j(n, d)$ by specifying $x_1, \ldots, x_{d-1}$, whereupon x_d is uniquely determined. So, each $\Gamma_j(n, d)$ has n^{d-1} vertices.

Consider the map

$$\varphi : \Gamma_0(n, d) \to \Gamma_j(n, d)$$

given by

$$(x_1, x_2, \ldots, x_d) \mapsto (x_1, x_2, \ldots, x_d + j),$$

where addition is performed modulo n. The claim is that this is a graph isomorphism. It is clearly a bijection. Moreover, for any $1 \le i < j \le d$,

$$(x_1, \ldots, x_i + k, \ldots, x_j - k, \ldots, x_d) \mapsto (x_1, \ldots, x_i + k, \ldots, x_j - k, \ldots, x_d + j),$$

so φ preserves adjacency. It follows that all the $\Gamma_j(n, d)$'s are isomorphic.

Lastly, fix two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ of the same value (say, 0). We construct a path connecting them. Suppose that $y_1 = x_1 + k$. Then we may step along an edge from $(x_1, \ldots, x_d)$ to $(y_1, x_2 - k, x_3, \ldots, x_d)$. By repeating this process, we end up

[5] Note that the resulting Cayley graphs are subgraphs (not induced) of the distance two graphs of the Hamming scheme $H(d, n)$, which have connection set $C = \bigcup_{1 \le i < j \le d} \sum_{1 \le k, \ell \le n-1} (0, \ldots, k, \ldots, \ell, \ldots, 0)$.

after $d-2$ steps at a vertex of the form $(y_1, y_2, \ldots, x'_{d-1}, x_d)$. But as the steps preserve values, we must have $x'_{d-1} + x_d = y_{d-1} + y_d$. Thus, there is an ℓ with $y_{d-1} = x'_{d-1} + \ell$ and $y_d = x_d - \ell$, in which case $(y_1, y_2, \ldots, x'_{d-1}, x_d) \sim (y_1, \ldots, y_d)$. It follows that $\Gamma_j(n,d)$ has diameter at most $d-1$. (In particular, $\Gamma_j(n,d)$ is connected, and $\Gamma(n,d)$ has exactly n connected components.) But the two vertices $(0, \ldots, 0)$ and $(1, 1, \ldots, 1, -(d-1))$ witness the fact that the diameter is at least $d-1$. $\square$

Proposition 2. *$\Gamma(n,d)$ and $T(n,d)$ are both regular of degree $(n-1)\binom{d}{2}$.*

Proof. Fix a vertex $(x_1, \ldots, x_d)$ of $\Gamma(n,d)$. Then for $1 \le i < j \le k$,

$$(x_1, \ldots, x_d) \sim (x_1, \ldots, x_i + k, \ldots, x_j - k, \ldots, x_d)$$

for $1 \le k \le n-1$. There are $\binom{d}{2}$ choices of (i,j) with $i < j$, and $n-1$ choices of k, giving a degree of $(n-1)\binom{d}{2}$. By Proposition 1, the graph $T(n,d)$ is an induced subgraph of $\Gamma(n,d)$, and constitutes its own connected component. Therefore it has the same degree as $\Gamma(n,d)$. $\square$

Let H be a finite group, and let $\mathfrak{S}_d$ denote the symmetric group on d letters. The *wreath product* $H \wr \mathfrak{S}_d$ is the group of elements of the form $(x_1, \ldots, x_d; \sigma)$ where $x_i \in H$ $(1 \le i \le d)$ and $\sigma \in \mathfrak{S}_d$, with product given by

$$(x_1, \ldots, x_d; \sigma)(y_1, \ldots, y_d; \tau) = (x_1 y_{\sigma^{-1}(1)}, \ldots, x_d y_{\sigma^{-1}(d)}; \sigma\tau).$$

Let $\operatorname{Aut}\Gamma$ denote the full automorphism group of a graph Γ.

Proposition 3. *The wreath product group $\mathbb{Z}_n \wr \mathfrak{S}_d$ is a subgroup of $\operatorname{Aut} T(n,d)$.*

Proof. The group $\mathbb{Z}_n \wr \mathfrak{S}_d$ acts on the vertices of $T(n,d)$ by

$$(x'_1, \ldots, x'_d) := (a_1, \ldots, a_d; \sigma) \circ (x_1, \ldots, x_d) = (a_1 + x_{\sigma^{-1}(1)}, \ldots, a_d + x_{\sigma^{-1}(d)}).$$

If $(y_1, \ldots, y_d) \sim (x_1, \ldots, x_d)$ then, for some pair (i,j) with $i < j$, $y_i - x_i = k$, $y_j - x_j = -k$, and $x_m = y_m$ for $m \ne i, j$. Suppose $\sigma^{-1}(i) = p$ and $\sigma^{-1}(j) = q$. Then $y'_p - x'_p = k$, $y'_q - x'_q = -k$, and $y'_m = x'_m$ for $m \ne p, q$, so adjacency is preserved under the action of the group. $\square$

Corollary 1. *The graph $T(n,d)$ is vertex transitive.*

Proof. Let $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ be two vertices of $T(n,d)$. Set $a_i := y_i - x_i$ for $1 \le i \le d$. Then

$$(y_1, \ldots, y_d) = (a_1, \ldots, a_d; 1) \circ (x_1, \ldots, x_d). \qquad \square$$

5 The spectrum of $T(n, d)$

The following result is well-known (see, *e.g.*, [8] or [16]).

Lemma 1. *Let A be the adjacency operator of a Cayley graph on an abelian group G with connection set C, and let ψ be a character of G. Then ψ is an eigenfunction of A with eigenvalue $\psi(C) := \sum_{c \in C} \psi(c)$.*

Proof. A character of G is a homomorphism from G to $\mathbb{C}$, so

$$(A\psi)(h) = \sum_{g \sim h} \psi(g) = \sum_{c \in C} \psi(ch) = \psi(C)\psi(h). \qquad \square$$

The characters of $\mathbb{Z}_n$ are given by

$$\chi_r(x) = e^{2\pi i r x/n} \qquad (0 \leq r \leq n-1).$$

Moreover, the characters of a Cartesian product of groups are just the products of the characters of each constituent. Hence, the characters of G are given by

$$\chi_{r_1,\ldots,r_d}(x_1, x_2, \ldots, x_d) = \prod_{\ell=0}^{d} e^{2\pi i r_\ell x_\ell/n} = \exp\left\{\sum_{\ell=0}^{d} 2\pi i r_\ell x_\ell/n\right\}, \tag{5}$$

where $(r_1, \ldots, r_d) \in [0, 1, \ldots, n-1]^d$.

Let $\lambda \vdash d$ be a partition of d. We write λ in two ways: as a nonincreasing sequence $(\lambda_1, \ldots, \lambda_\ell)$ of length ℓ with $\sum_i \lambda_i = d$, and in multiplicity notation $\lambda = 1^{m_1} \cdots d^{m_d}$, where m_i is the number of occurrences of i in the partition. Also, recall the falling factorial symbol:

$$(x)_m := x(x-1)\cdots(x-m+1)$$

This brings us to the main result.

Theorem 1. *The eigenvalues of the simplicial torus rook graph $T(n, d)$ are labeled by integer partitions $\lambda \vdash d$ and are given by*

$$n \sum_{i=1}^{\ell} \binom{\lambda_i}{2} - \binom{d}{2},$$

each with multiplicity

$$m(\lambda) = \frac{d!}{\lambda_1!\lambda_2!\ldots,\lambda_\ell!} \frac{(n)_{m_1+m_2+\cdots+m_d}}{m_1!m_2!\cdots m_d!n}.$$

Proof. By Proposition 1, the eigenvalues of $\Gamma(n,d)$ are given by

$$\eta_{r_1,\ldots,r_d} = \sum_{1\le i<j\le d}\sum_{k=1}^{n-1} e^{2\pi i k(r_i-r_j)/n}.$$

But $\sum_{k=1}^{n-1} e^{2\pi i k r/n} = n\delta_{r,0}-1$, where $\delta_{r,s}$ is the Kronecker delta, so

$$\eta_{r_1,\ldots,r_d} = n\sum_{1\le i<j\le d}\delta_{r_i,r_j} - \binom{d}{2}. \tag{6}$$

Evidently, the eigenvalues depend only on the number of pairs of indices $(r_1,\ldots,r_d)$ that are equal. To compute these numbers, we proceed as follows. First, choose a partition $\lambda \vdash d$. We want to distribute the parts of this partition of d into n boxes. To each such distribution (a weak composition of d into n parts) we may associate a number of sequences $(r_1,\ldots,r_d)$, in a way to be described. Then we must multiply this number by the number of pairs of indices in these sequences that are equal.

By way of illustration, consider the case $n=5$ and $d=3$. The three partitions of 3 are (3), $(2,1)$, and $(1,1,1)$. Suppose $\lambda=(2,1)$. A few of the distributions of the parts of this partition into 5 boxes are $[2,1,0,0,0]$, $[2,0,1,0,0]$, $[0,1,2,0,0]$, and so on. In this case, we will have $5\cdot 4 = 20$ such distributions. Next, we view the distributions as multiplicities of the numbers appearing in a sequence. Each list of multiplicities gives rise to a permuted set of sequences. For instance, the distribution $[2,0,1,0,0]$ can be thought of as representing all sequences of length 3 containing two 1's and one 3, namely, $(1,1,3)$ $(1,3,1)$ and $(3,1,1)$ (these are counted by $\binom{d}{\lambda_1,\ldots,\lambda_\ell}=\binom{3}{2,1}=3$). We obtain all $n^d=125$ sequences of the form $(r_1,\ldots,r_d)$ in this way, because by the multinomial theorem

$$n^d = \sum_{a_1+\cdots+a_n=d,\, a_i\ge 0}\binom{d}{a_1,a_2,\ldots,a_n}.$$

Each of the sequences $(1,1,3)$ $(1,3,1)$ and $(3,1,1)$ has one pair of indices that are equal, so for each of these, $\sum_{1\le i<j\le d}\delta_{r_i,r_j}=1$. Hence, the eigenvalue associated to $\lambda=(2,1)$ is $n\cdot 1-\binom{d}{2}$, with multiplicity $20\cdot 3=60$.

In general, given a partition $\lambda=(\lambda_1,\ldots,\lambda_\ell)\vdash d$, written also in multiplicity notation as $1^{m_1}2^{m_2}\cdots d^{m_d}$, there are

$$\binom{n}{m_1}\binom{n-m_1}{m_2}\cdots\binom{n-m_1-\cdots-m_{d-1}}{m_d} = \frac{(n)_{m_1+\cdots+m_d}}{m_1!m_2!\cdots m_d!}$$

partition	eigenvalue	multiplicity
$3 = 3^1$	$3n-3$	1
$21 = 1^1 2^1$	$n-3$	$3(n-1)$
$111 = 1^3$	-3	$(n-1)(n-2)$

Table 1: The spectrum of $T(n,3)$

partition	eigenvalue	multiplicity
$4 = 4^1$	$6n-6$	1
$31 = 1^1 3^1$	$3n-6$	$4(n-1)$
$22 = 2^2$	$2n-6$	$3(n-1)$
$211 = 1^2 2^1$	$n-6$	$6(n-1)(n-2)$
$1111 = 1^4$	-6	$(n-1)(n-2)(n-3)$

Table 2: The spectrum of $T(n,4)$

ways to place the parts $\{\lambda_1, \ldots, \lambda_\ell\}$ into n boxes. To each such distribution we construct $\binom{d}{\lambda_1,\ldots,\lambda_d}$ sequences of the form $(r_1, \ldots, r_d)$ in the manner above. For each such sequence

$$\sum_{1 \le i < j \le d} \delta_{r_i, r_j} = \sum_{i=1}^{\ell} \binom{\lambda_i}{2}.$$

Combining this with (6), we see that the eigenvalues of $\Gamma(n,d)$ are indexed by partitions $\lambda \vdash d$ and are given by

$$n \sum_{i=1}^{\ell} \binom{\lambda_i}{2} - \binom{d}{2},$$

each with multiplicity

$$k(\lambda) := \frac{(n)_{m_1+\cdots+m_d}}{m_1! m_2! \cdots m_d!} \binom{d}{\lambda_1, \ldots, \lambda_d}.$$

By Proposition 1, $\Gamma(n,d)$ is comprised of n identical copies of $T(n,d)$, so the eigenvalues of $T(n,d)$ and $\Gamma(n,d)$ are the same, but the multiplicities of the eigenvalues of $T(n,d)$ are just $k(\lambda)/n$. $\square$

The theorem is illustrated in Tables 1 and 2 for $d = 3$ and $d = 4$, respectively,

Remark 1. Evidently the spectrum of $T(n,d)$ is integral. In fact, integrality is a consequence of a general result of Bridges and Mena [2] (see also [6] and [10]). Bridges and Mena proved that, if the connection set C is contained in the Boolean algebra generated by all the subgroups of an abelian group G, then the Cayley graph $X(G,C)$ is integral. In our case, for fixed i and j, the group elements $(0,\ldots,\underbrace{k}_{i},\ldots,\underbrace{-k}_{j},\ldots,0)$ for $1 \le k \le n-1$ are contained in the subgroup of $\mathbb{Z}_n^d$ given by these elements together with the zero element.

Remark 2. The quantity $b(\lambda) = \sum_i \binom{\lambda_i}{2}$ appearing in Theorem 1 appears often in the context of symmetric functions and the representation theory of the symmetric group (*e.g.*, [11], Ch. 1, [15], p. 374, [7], p. 40). For instance, if T is the set of transpositions of $\mathfrak{S}_d$ and χ^λ is the irreducible character of $\mathfrak{S}_d$ associated to λ, we have

$$\frac{\chi^\lambda(T)}{\chi^\lambda(1)} = \frac{1}{\chi^\lambda(1)} \sum_{t \in T} \chi^\lambda(t) = b(\lambda) - b(\lambda'),$$

where λ' is the conjugate partition associated to λ. It is perhaps not too surprising that $b(\lambda)$ shows up when computing the spectrum of the graphs $\Gamma(n,d)$ and $T(n,d)$, as both graphs admit $\mathfrak{S}_d$ as a group of automorphisms.

Remark 3. As mentioned in the introduction, it was shown in [3] that the least eigenvalue of $S(n,d)$ is equal to $\max\left\{-n, -\binom{d}{2}\right\}$. In our case, the least eigenvalue of $T(n,d)$ is easily seen to be $-\binom{d}{2}$ for all $n > 0$.

6 The special case $d = 3$

Closer examination of Table 1 reveals that $T(n,3)$ has exactly three eigenvalues. We call $3(n-1)$ the 'trivial' eigenvalue, because every regular graph possesses an eigenvalue equal to its degree, which by Proposition 2 is $3(n-1)$. We observe that the other two eigenvalues, namely $n-3$ and -3 are distinct. Therefore, by a general result ([4] Theorem 9.1.2) or ([9], Lemma 10.2.1), $T(n,3)$ must be strongly regular. (Recall that a graph Γ is said to be *strongly regular* with parameters (v,k,λ,μ) if it is has v vertices, is regular of degree k, and the number of common neighbors of two vertices x and y is λ or μ, according as x and y are adjacent or not.) We can see this directly as well.

Proposition 4. *The graphs $T(n,3)$ are strongly regular with parameters $(n^2, 3(n-1), n, 6)$.*

Proof. By Proposition 1, $T(n,d)$ has n^{d-1} vertices. We have already observed that $k = 3(n-1)$. Now choose two adjacent vertices of $T(n,d)$. By vertex transitivity, we may choose the first vertex to be $u = (0,0,0)$ and the second vertex to be, say, $v = (0,1,n-1) = (0,1,-1)$. Clearly, the $n-2$ vertices of type $(0,k,-k)$ with $k \neq 0,1$ are adjacent to both u and v. To this collection we must add $(1,0,-1)$ and $(-1,1,0)$, so that $\lambda = n$. On the other hand, if we fix one vertex to be u, a non-neighbor of u is of the form $v = (a,b,c)$ with $a+b+c = 0$ and a, b and c are all nonzero. (Repetitions are allowed.) Neighbors of v that are also adjacent to u must have 0 as one of its entries. So there are six possibilities: $(0, b+a, c) = (0, -c, c)$, $(0, b, c+a) = (0, b, -b)$, $(a+b, 0, c) = (-c, 0, c)$, $(a, 0, b+c) = (a, 0, -a)$, $(a+c, b, 0) = (-b, b, 0)$, and $(a, b+c, 0) = (a, -a, 0)$. These are all necessarily distinct. For instance, we cannot have $(0,-c,c) = (0,b,-b)$, else $b = -c$, implying that $a = 0$, a contradiction. The same argument works for all other pairs. In particular, $\mu = 6$.

We can obtain λ and μ in another way. According to ([4], Theorem 9.1.3) we have

$$rs = \mu - k \quad \text{and} \quad r + s = \lambda - \mu,$$

where r and s are the two non-trivial eigenvalues of a strongly regular graph. In our case we have $r = n-3$ and $s = -3$, so

$$\mu = k + rs = 3(n-1) - 3(n-3) = 6 \quad \text{and} \quad \lambda = r + s + \mu = n. \qquad \square$$

An *orthogonal array* with parameters d and n is a d by n^2 array with entries chosen from $[n]$ such that, for any two rows of the array, the (ordered) vertical pairs are all distinct. For instance, here is an orthogonal array with parameters $d = 3$ and $n = 4$:

$$\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\
1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4. \\
1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1
\end{array}$$

When $d = 3$ this notion is equivalent to that of a Latin square. To see why, label the rows of the above array $\boldsymbol{r}$, $\boldsymbol{c}$, and $\boldsymbol{s}$ (for 'row', 'column', and 'symbol'). Reading off the elements of the i^{th} column, we place $s(i)$ at $(r(i), c(i))$ in a square of size n. The defining condition of an orthogonal array guarantees that the resulting square is a Latin square.

When $d > 3$ an orthogonal array is equivalent to a set of $d-2$ mutually orthogonal Latin squares ([17], Ch. 17).

Given an orthogonal array with parameters n and d, we can form a graph, labeled $\mathrm{OA}(n,d)$, whose vertices are the columns of the array, two such columns being adjacent if they agree in exactly one coordinate position. For instance, in the above array, the first two columns would be adjacent in $\mathrm{OA}(4,3)$ because they agree only in the first coordinate. When $d = 3$, this is the same thing as a Latin square graph (equivalently, an ordinary rook graph): given a Latin square array of size n, form a graph on the positions of the array, where two positions (i,j) and (k,ℓ) are adjacent if $i = k$ or $j = \ell$ or (i,j) and (k,ℓ) contain the same symbol. Note that, by results of Denes and Keedwell [5] and Phelps [14], different Latin squares of the same size can give rise to non-isomorphic Latin square graphs.

By Theorem 10.4.2 in [9], the graphs $\mathrm{OA}(n,d)$ (written there as $\mathrm{OA}(d,n)$) are strongly regular, with parameters $(n^2, d(n-1), n-2+(d-1)(d-2), d(d-1))$; for $d = 3$ the parameters become $(n^2, 3(n-1), n, 6)$. Comparing this to the conclusion of Proposition 4, we see that $T(n,3)$ and $\mathrm{OA}(n,3)$ are both strongly regular with the same parameter set. This suggests that $T(n,3)$ is isomorphic to some $\mathrm{OA}(n,3)$.

Proposition 5. *Let L be the Latin square arising as the negative of the group multiplication table of $\mathbb{Z}_n$. Then the Latin square graph arising from L is isomorphic to $T(n,3)$.*

Proof. The entries of L are given by

$$L_{ij} = -(i+j) \bmod n.$$

which is clearly a (circulant) Latin square. The columns of the corresponding orthogonal array sum to zero, and are therefore also vertices of $T(n,3)$. Moreover, the condition for adjacency in the Latin square graph and $T(n,3)$ are the same. □

7 Some questions

In [3], the authors deduce several other properties of the simplicial rook graphs, such as their automorphism group and local structure. One could try to do something similar in the case of abelian rook graphs. For instance, by Proposition 3, $T(n,d)$ admits the wreath product group $\mathbb{Z}_n \wr \mathfrak{S}_d$ as a group of automorphisms. Is this the full automorphism group? Do the abelian rook graphs defined here tell us anything useful about simplicial torus graphs?

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