Totally Symmetric Equiangular Line Sets

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Abstract

We offer an elementary construction of some highly symmetric equiangular line sets.

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1 Introduction

A finite collection $\mathcal{L}$ of lines through the origin in $\mathbb{R}^d$ is said to be *equiangular* provided the (lesser) angle between any two lines in $\mathcal{L}$ is constant. The simplest nontrivial example of an equiangular line set in $\mathbb{R}^2$ consists of the diagonals of a regular hexagon, while in $\mathbb{R}^3$ we have the four diagonals of the cube and the six diagonals of the icosahedron. In addition to their obvious aesthetic appeal, equiangular line sets have deep connections to combinatorics, design theory, coding theory, communication theory, and many other areas of mathematics and physics, and their study is a very active area of research. See [5] (also [6], Chapter 11) for a nice introduction and overview. For some recent results, see, *e.g.*, [1, 2, 3, 8, 14].

The main problem in this area is to determine $N(d)$, the maximum number of equiangular lines in dimension d, and if possible, construct them. Although many specific instances of $N(d)$ are known, a complete characterization seems entirely out of reach. In general we have the following two uniform bounds. The first is known as the *absolute bound*.

Theorem 1 (Gerzon [9])**.** *For $d \geq 2$ we have*

$$N(d) \leq \binom{d+1}{2}.$$

If equality holds then $d = 2$, $d = 3$, or $d + 2$ is the square of an odd integer.

The second is called the *relative bound* and requires a bit more discussion. Let $\Phi \subset \mathbb{R}^d$ be a set of n distinct vectors whose linear spans are the lines $\mathcal{L}$. The line set $\mathcal{L}$ is equiangular provided there exists a constant $\alpha < 1$ such that $|(v, w)/\sqrt{(v, v)(w, w)}| = \alpha$ for all $v, w \in \Phi$ with $v \neq w$. Here $(\cdot, \cdot)$ denotes the usual inner product on $\mathbb{R}^d$. The constant α is the *angle* of $\mathcal{L}$. By a result of Neumann [9], if $n > 2d$ then $1/\alpha$ must be an odd integer. The relative bound is as follows.

d	2	3	4	5	6	7-13
$N(d)$	3	6	6	10	16	28
α^{-1}	2	$\sqrt{5}$	$\sqrt{5}$,3	3	3	3

Table 1: The maximum number of equiangular lines in $\mathbb{R}^d$ and their angles

Theorem 2 (Lemmens & Seidel [9]). *If $\mathcal{L}$ is a set of n equiangular lines in $\mathbb{R}^d$ with angle α then*

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

For small values of d we have the results in Table 1 ([5]). As of this writing, the first unknown value is $N(14)$, where the correct answer is either 28 or 29. Examination of Table 1 shows that the absolute bound is achieved only in dimensions 2, 3, and 7, consistent with Theorem 1. [1] Also, the relative bound is met in dimensions 2, 3, 5, 6, and 7.

Constructing equiangular line sets is difficult and, apart from some already well-known configurations, mostly a matter of cleverness and luck. In this paper we discuss an elementary construction of some equiangular line sets in dimensions $d \leq 8$. This will yield a well-known line set realizing the absolute bound, and will rule out a large class of line sets in all dimensions.

2 Totally symmetric equiangular line sets

Equiangular lines are highly symmetric structures, so it makes sense to look for sets admitting some kind of symmetry group. [2] The simplest such group is

[1]There is only one more dimension known in which the absolute bound is achieved, namely $d = 23$. The corresponding equiangular line set is related to the famous Witt design and the celebrated binary Golay code. For more on this, see, *e.g.*, [6, 7, 12].

[2]Most of the well-known equiangular line sets meeting the relative or absolute bound have large symmetry groups. But in general it is difficult to construct such a set given

common b's	inner product
0	$4ab + (d-4)a^2$
1	$2ab + b^2 + (d-3)a^2$
2	$2b^2 + (d-2)a^2$

Table 2: Possible inner products for permutations of $(b, b, a, \ldots, a)$

$\mathfrak{S}_d$, the permutation group on d elements. So, we fix a vector $v = (v_1, \ldots, v_n)$ and consider its orbit under the defining representation of the group, namely

$$\Phi = \{(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) : \sigma \in \mathfrak{S}_d\}.$$

We call such a set (or its corresponding line set) *totally symmetric.* When is such a set equiangular, if ever?

In what follows we consider various types of generating vectors v. For example, the case $v = (a, \ldots, a)$ is obviously trivial, as there is only one vector in the orbit. The next simplest type is $(b, a, \ldots, a)$ in which all the entries of v are equal save one. But this case is uninteresting, because the inner product between any two such vectors is obviously constant. In particular, we have found d equiangular lines in $\mathbb{R}^d$. The standard basis consisting of all vectors of the form $(0, \ldots, 1 \ldots, 0)$ is a simple example in which $\alpha = \pi/2$.

Things become much more interesting next. Assume $v = (b, b, a, \ldots, a)$. Now we have three cases for the inner product (v, w) depending on whether the b's overlap in zero, one, or two places. The possible inner products are classified in Table 2.

If the two b entries coincide in v and w then $v = w$, so we may exclude the last case. Therefore we have only two possibilities in order that the inner products agree up to sign:

$$4ab + (d-4)a^2 = 2ab + b^2 + (d-3)a^2 \qquad (1)$$

only the group. For a discussion from the point of view of finite frames, see, for example, [?].

or

$$4ab + (d-4)a^2 = -(2ab + b^2 + (d-3)a^2). \tag{2}$$

Equation (1) gives

$$2ab = b^2 + a^2 \Rightarrow (a-b)^2 = 0 \Rightarrow a = b,$$

which puts us back into the trivial case. Equation (2) gives

$$6ab + b^2 + (2d-7)a^2 = 0. \tag{3}$$

If $a = 0$ then $b = 0$, which is trivial. If $b = 0$ then either $a = 0$ (again, trivial), or else $d = 7/2$, an impossibility. So without loss of generality we may scale the vectors so that $b = 1$. This yields

$$(2d-7)a^2 + 6a + 1 = 0.$$

We get a real solution provided the discriminant is nonnegative, which gives

$$36 - 8d + 28 \geq 0 \Rightarrow d \leq 8.$$

Apparently there are no other totally symmetric line sets of type $(b, b, a, \ldots, a)$ in dimensions greater than 8. Already we have learned something interesting!

Now suppose $d = 8$. Then we have

$$9a^2 + 6a + 1 \Rightarrow a = \frac{1}{9}(-6 \pm \sqrt{36 - 36}) = -\frac{1}{3}.$$

How many such vectors are there? We can place the b's in $\binom{d}{2} = \binom{8}{2} = 28$ different ways, so $n = 28$. We have therefore found 28 equiangular lines in $\mathbb{R}^8$. Up to a scale factor, these vectors are all permutations of $(3, 3, -1, -1, -1, -1, -1, -1)$. Evidently the entries sum to zero, so in fact these lines actually live in a seven dimensional subspace. In particular, we have found 28 equiangular lines in $\mathbb{R}^7$, and this number meets the absolute bound. Classically this configuration of lines can be obtained as the

d	2	3	4	5	6	7	8
a	$1 \pm \frac{2}{\sqrt{3}}$	$3 \pm \sqrt{10}$	$-3 \pm 2\sqrt{2}$	$-1 \pm \frac{\sqrt{6}}{3}$	$-\frac{1}{5}, -1$	$\frac{-3 \pm \sqrt{2}}{7}$	$-\frac{1}{3}$
$\|\Phi\|$	1	3	6	10	15	21	28
α^{-1}	–	3	3	3	3	3	3

Table 3: Allowed parameters for a subclass of totally symmetric equiangular line sets by dimension

diagonals of a three dimensional polytope in $\mathbb{R}^7$ called Gosset's semiregular polytope 3_{21} (in Coxeter's notation—see [4]). This, in turn, is related to the Lie group E_7 and other mathematical structures.

What happens in lower dimensions? Plugging numbers into (3) and solving gives the allowed values of a (again assuming $b = 1$), the inverse angles of the corresponding equiangular line sets, and their set sizes (namely $\binom{d}{2}$). The results are given in Table 3. Comparison with Table 1 reveals that the totally symmetric line sets of type $(b, b, a, \ldots, a)$ are maximal in dimensions 4, 5, and 8. We have thus obtained some very interesting equiangular line configurations with minimal effort.

3 More totally symmetric line sets?

Now we consider the next simplest case, namely the one in which b appears with multiplicity three, so that Φ consists of all permutations of $(b, b, b, a, \ldots, a)$. By considering the possible inner products as before we obtain the results in Table 4.

But now we discover an interesting phenomenon. As before we may set $b = 1$ without loss of generality. For brevity let (j) denote the j^{th} entry in

common b's	inner product
0	$6ab + (d-6)a^2$
1	$b^2 + 4ab + (d-5)a^2$
2	$2b^2 + 2ab + (d-4)a^2$
3	$3b^2 + (d-3)a^2$

Table 4: Possible inner products for permutations of $(b, b, b, a, \ldots, a)$

the table. If we equate (0) and (1) we get

$$6a + (d-6)a^2 = 1 + 4a + (d-5)a^2,$$

which yields the trivial case $a = 1$. So we are forced to conclude that (0) and (1) are opposite in sign. If (0) and (2) are the same sign we get another triviality (check!), but if they are opposite in sign, then (1) and (2) must be the same sign, and again we get a triviality. A similar argument works for any multiplicity of b greater than three. Therefore we conclude that there are no more totally symmetric equiangular line sets with vectors containing only two distinct entries!

4 A rearrangement inequality

In fact, more is true.

Theorem 3. *Let Φ be a totally symmetric equiangular line set generated by a vector $v = (v_1, \ldots, v_n)$. Then v must be one of the following types: (i) $v = (a, \ldots, a)$, (ii) $v = (b, a, \ldots, a)$, or (iii) $v = (b, b, a, \ldots, a)$. In particular, apart from the uninteresting cases (i) and (ii), every such line set is one of the types given in Table 3.*

Proof. By virtue of the preceding discussion we may assume v has at least three distinct entries. Without loss of generality we may assume v to be of the form (a, b, c, u), where $a < b < c$ and u is any vector of size $d - 3$. The possible inner products between distinct vectors are all of the form $(v, \sigma v) := \sum_{k=1}^{d} v_i v_{\sigma(i)}$, where $\sigma \neq e$, the identity permutation.

Consider the permutations $\sigma_1 = (1, 3, 2, \pi)$, $\sigma_2 = (3, 1, 2, \pi)$ and $\sigma_3 = (3, 2, 1, \pi)$, where π is any permutation of $\{4, \ldots, d\}$ (or empty if $d = 3$). We claim that we must have $(v, \sigma_i v) \neq (v, \sigma_j v)$ for any $i, j \in \{1, 2, 3\}$ with $i \neq j$, which proves the theorem, as we cannot have $(v, \sigma_i v) = -(v, \sigma_j v)$ for all $i, j \in \{1, 2, 3\}$ either.

To prove the claim, we follow the approach used in ([10], Chapter 5) to prove the celebrated rearrangement inequality. [3] First recall that an *inversion* of a permutation $\sigma \in \mathfrak{S}_d$ is a pair (j, k) such that $\sigma(j) > \sigma(k)$, and let $\ell(\sigma)$ denote the number of such pairs. Suppose σ has an inversion (j, k), and define $\tau \in S_d$ by (i) $\tau(i) = \sigma(i)$ if $i \notin \{j, k\}$, (ii) $\tau(j) = \sigma(k)$, and (iii) $\tau(k) = \sigma(j)$. Then $\ell(\tau) < \ell(\sigma)$, and a bit of algebra shows that

$$(v, \tau v) - (v, \sigma v) = (v_k - v_j)(v_{\tau(k)} - v_{\tau(j)}).$$

Observing that $\ell(\sigma_3) > \ell(\sigma_2) > \ell(\sigma_1)$ gives

$$(v, \sigma_1 v) > (v, \sigma_2 v) > (v, \sigma_3 v).$$

□

5 Beyond the symmetric group

Evidently totally symmetric equiangular line sets are rare. So it makes sense to consider equiangular line sets arising from the orbits of some vector under

[3] The *rearrangement inequality* states that, if $-\infty < a_1 \leq a_2 \leq \cdots \leq a_d < \infty$ and $-\infty < b_1 \leq b_2 \leq \cdots \leq b_d < \infty$ are any two sequences, then for any permutation $\sigma \in \mathfrak{S}_d$ we have

$$\sum_{k=1}^{d} a_k b_{d-k+1} \leq \sum_{k=1}^{d} a_k b_{\sigma(k)} \leq \sum_{k=1}^{d} a_k b_k.$$

other representations of $\mathfrak{S}_d$ or else other groups G having d dimensional representations. But our previous considerations place stringent restrictions on the allowed groups and representations. In particular, considerations such as those given above rule out many classes of permutation representations.

Equiangular line sets are related to many beautiful and important mathematical structures. This vast territory has been studied extensively because of its connections to classical areas of mathematics such as sphere packings, coding theory, and finite simple groups [12]. But much still remains to be explored, and more surprises surely await discovery.

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