Kronecker coefficients for three hook shapes

Paul Renteln

Department of Physics
California State University
San Bernardino, CA 92407
prenteln@csusb.edu

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Abstract

We derive a simple geometric characterization of the Kronecker coefficients $\langle \chi_j \chi_k, \chi_\ell \rangle$, where χ_k is the character of the irreducible $\mathfrak{S}_n$ module $V^{(n-k,1^k)}$.

Keywords: symmetric group, Kronecker coefficients, hook shapes, cycle index

1 Introduction

Let χ^λ be the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by the partition $\lambda \vdash n$. [1] The *Kronecker coefficients* $g_{\mu\nu\lambda}$ are defined by

$$\chi^\mu \chi^\nu = \sum_\lambda g_{\mu\nu\lambda} \chi^\lambda.$$

Equivalently, $g_{\mu\nu\lambda} = \langle \chi^\mu \chi^\nu, \chi^\lambda \rangle$, where $\langle \chi, \psi \rangle$ is the usual inner product on characters [2]

$$\langle \chi, \psi \rangle = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \chi(\pi)\psi(\pi).$$

[1] To avoid trivialities, we assume $n \geq 2$.

[2] In this expression we have used the well-known fact that the characters of the symmetric group are real.

From this expression, two striking facts emerge. First, the Kronecker coefficients are symmetric in $\{\mu, \nu, \lambda\}$. Second, because the product of two characters is again a character [3], the Kronecker coefficients are *nonnegative*. Murnaghan [16, 17, 18] studied these coefficients and challenged his readers to find a combinatorial interpretation for them, a challenge that remains unanswered, except in various special cases. [4]

There is another expression for the Kronecker coefficients, developed first by Littlewood [12] using the theory of symmetric functions. [5] Employing the characteristic map of Frobenius, Littlewood showed that

$$s_\mu * s_\nu = \sum_\lambda g_{\mu,\nu,\lambda} s_\lambda,$$

where s_λ denotes the Schur function indexed by λ, and '$*$' denotes the *internal product*. The comultiplication formula for Kronecker coefficients is

$$s_\lambda[XY] = \sum_{\mu,\nu} g_{\mu\nu\lambda} s_\mu[X] s_\nu[Y],$$

where $X = x_1 + \cdots + x_n$ and $Y = y_1 + \cdots + y_n$ denote two alphabets (written additively), $XY = x_1y_1 + x_1y_2 + \cdots + x_1y_n + x_2y_1 + \cdots + x_ny_n$, and $s_\lambda[X] = s_\lambda(x_1, \ldots, x_n)$.

These last two algebraic definitions employ the notion of *plethysm* (a term coined by Littlewood), which is notoriously difficult to handle. Indeed, it seems fair to say that the difficulty of computing Kronecker coefficients rests with the apparent intractability of plethystic computations. Nevertheless, when the partitions involved are relatively simple it is possible to extract some information. For instance, Rosas [25] used the comultiplication formula to compute the coefficients $g_{\mu\nu\lambda}$ when μ and ν are either hooks or two-rowed partitions. [6] These results were extended by Blasiak [4] and Liu [13] when only one of the partitions is a hook shape and the others are arbitrary. Recently, Ashraf [2], using combinatorial methods together with the generating function for hook characters [7], was able to obtain a simple generating function for the reduced Kronecker coefficients [8] for

[3] It is in fact the tensor product character.

[4] The literature on Kronecker coefficients is vast. See, *e.g.,* [19] and references therein.

[5] For all undefined terms, see [14] or [26].

[6] Some of these coefficients had been computed earlier by other means by Lascoux [11], Remmel [23], and Remmel and Whitehead [24].

[7] Ashraf attributes this result to Goupil [9], but the generating function for hook characters goes back to Aitken [1]. See Stanley [26], Ex. 7.72. See also below.

[8] For the definition, see, e.g., [17].

hook shapes, thereby reproducing a result of Rosas. [9] In fact, his proof yields a generating function for the coefficients $g_{\mu\nu\lambda}$ when all three partitions are hooks.

The purpose of this paper is to compute the Kronecker coefficients corresponding to three hook shapes using only elementary algebraic means (together with a few basics concerning the representation theory of the symmetric group and some rudimentary symmetric function theory), and to present the results in a novel geometric form. Although some of our analysis resembles that of Ashraf, we avoid subtle combinatorics, and we express our final result in terms of symmetric functions rather than in terms of a generating function. This yields a nice geometric characterization of the Kronecker coefficients in three hook case in terms of integer points in triangles.

2 A generating function for hook characters

We first recall a basic fact about characters of exterior product representations. Let U be an m-dimensional representation of a finite group G. If the eigenvalues of g in the representation U are $\{\eta_1, \ldots, \eta_m\}$, then the eigenvalues of g in $\wedge^k U$ are of the form $\{\eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}\}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. Their sum is thus $e_k(\lambda_1, \ldots, \lambda_m)$, where e_k is the k^{th} elementary symmetric function. Hence,

$$p_g^U(x) := \det\big|_U(1 - xg) = \prod_i (1 - \lambda_i x) = \sum_{k=0}^{n} (-1)^k (\mathrm{Tr}_{\wedge^k U}\, g) x^k = \sum_{k=0}^{n} (-1)^k \chi_k^U(g) x^k, \quad (1)$$

where $\chi_k^U(g)$ is the character of the representation $\wedge^k U$ evaluated at g.

Let D be an n-dimensional vector space with standard basis $\{e_i\}$. Then D affords the *defining representation* of $\mathfrak{S}_n$, given by $\pi e_i = e_{\pi^{-1}(i)}$ for $\pi \in \mathfrak{S}_n$. Let V^λ denote the irreducible module of $\mathfrak{S}_n$ indexed by the partition λ. As is well-known, we have

$$D = V^{(n)} \oplus V^{(n-1,1)}.$$

Inspired by the theory of Coxeter groups, $V^{(n-1,1)}$ is often called the *reflection representation.* [10] It was Aitken [1] who first showed that the exterior powers of $V^{(n-1,1)}$ coincide

[9] See also [6].

[10] It is so-named because $\mathfrak{S}_n$ is the Coxeter group of type A_{n-1}: the basis elements of $V^{(n-1,1)}$ may be chosen to be the $n-1$ vectors $\{e_i - e_{i+1}\}_{1\leq i\leq n-1}$, corresponding to the reflection symmetries of a simplex.

with the irreducible representations associated to partitions of hook shape. Specifically, he proved that

$$\wedge^k V^{(n-1,1)} \cong V^{(n-k,1^k)} \quad (0 \leq k \leq n-1). \tag{2}$$

His proof used generating functions together with the Frobenius formula for the irreducible characters of $\mathfrak{S}_n$. [11] We may exploit this isomorphism to obtain a simple generating function for the hook characters.

In the defining representation D of $\mathfrak{S}_n$, the permutation π is represented by a permutation matrix $P(\pi)$. Every permutation admits a disjoint cycle decomposition. Thus, every permutation matrix can be written as a direct sum of cyclic permutation matrices. Every cyclic permutation matrix for a cycle w of size k has characteristic polynomial $\det(1-xw) = 1-x^k$. So, for a permutation π with cycle type $\mu = 1^{\mu_1} 2^{\mu_2} \cdots t^{\mu_\ell}$ (where μ^j is the number of j cycles) the characteristic polynomial of π in the representation D is

$$p_\pi^D(x) = (1-x)^{\mu_1}(1-x^2)^{\mu_2} \cdots (1-x^\ell)^{\mu_\ell}.$$

Abbreviating $p_\pi^{V^{(n-1,1)}}(x)$ simply by $p_\pi(x)$, it follows that [12]

$$p_\pi(x) = (1-x)^{\mu_1 - 1}(1-x^2)^{\mu_2} \cdots (1-x^\ell)^{\mu_\ell}. \tag{3}$$

Combining (1), (2), and (3), we conclude that

$$\chi_k(\pi) = [(-x)^k] p_\pi(x).$$

where χ_k is the character of $V^{(n-k,1^k)}$.

3 Kronecker coefficients of hooks

Next, we turn to a computation of the Kronecker coefficients

$$g_{jk\ell} := \langle \chi_j \chi_k, \chi_\ell \rangle.$$

[11] For other proofs of this relation, see [26], Ex. 7.72, [7], Proposition 5.4.12, and [8], Proposition 5.1. We remark that, as a consequence of Aitken's proof, one discovers the non-obvious fact that the representations $\wedge^k V^{(n-1,1)}$ are irreducible. Actually, this fact is a special case of a more general result holding for all Coxeter groups (of which the symmetric group is an instance), namely that the exterior powers of the reflection representation of any Coxeter group are irreducible. This more general result was first observed by Steinberg. See [5], Ch. 5 Ex. §2.3, [10], Theorem 24-3A, and [7], Theorem 5.1.4.

[12] This is a consequence of $p_\pi^D(x) = p_\pi^{(n)}(x) p_\pi^{(n-1,1)}(x)$, which follows from the direct sum decomposition of D, and $p^{(n)}(x) = 1-x$, which follows from the fact that $V^{(n)}$ is the trivial representation.

Using (2) we can write

$$g_{jk\ell} = [(-1)^{j+k+\ell} x^j y^k z^\ell] \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} p_\pi(x) p_\pi(y) p_\pi(z).$$

To evaluate this, we appeal to the cycle index. Recall that the cycle index of the symmetric group $\mathfrak{S}_n$ is

$$Z_n(\mathbf{u}) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} u_1^{\mu_1(\pi)} u_2^{\mu_2(\pi)} \cdots u_n^{\mu_n(\pi)},$$

where $\mu_j(\pi)$ is the number of cycles of length j in the disjoint cycle decomposition of π. Thus,

$$g_{jk\ell} = [(-1)^{j+k+\ell} x^j y^k z^\ell] \; u_1^{-1} Z_n(\mathbf{u})\big|_{u_i = (1-x^i)(1-y^i)(1-z^i)}.$$

Recall also ([26], p.xxx) that the generating function for Z_n is

$$\sum_n Z_n(\mathbf{u}) t^n = \exp\left\{ \sum_{i \geq 1} \frac{u_i t^i}{i} \right\}.$$

Hence, using the expansion

$$\log(1-x) = -\sum_{i \geq 1} \frac{x^i}{i}$$

and suppressing the dependence of $g_{jk\ell}$ on n, we obtain [13]

$$\sum_n g_{jk\ell} t^n$$
$$= [(-1)^{j+k+\ell} x^j y^k z^\ell] \frac{(1-xt)(1-yt)(1-zt)(1-xyzt)}{(1-t)(1-x)(1-y)(1-z)(1-xyt)(1-xzt)(1-yzt)}. \quad (4)$$

We now proceed to expand the right hand side of (4). Evidently, it is symmetric in x, y, and z, so it admits an expansion in terms of symmetric functions in those variables. We shall find such an expansion in terms of Schur functions.

Using partial fractions we can write

$$\frac{1}{(1-xyt)(1-xzt)(1-yzt)} = \sum_a w_a t^a,$$

[13] Compare [2], proof of theorem 3.7.

where

$$w_a := A(xy)^a + B(yz)^a + C(xz)^a,$$

and where we have defined

$$A = \frac{xy(x-y)}{\Delta}, \quad B = \frac{yz(y-z)}{\Delta}, \quad C = \frac{xz(z-x)}{\Delta},$$

and

$$\Delta := (x-y)(y-z)(x-z).$$

Also,

$$(1-xt)(1-yt)(1-zt)(1-xyzt) = \sum_b (-1)^b e_b(E)\, t^b,$$

where $E = (x, y, z, xyz)$. Thus, the right hand side of (4) becomes

$$\begin{aligned} RHS &= \frac{1}{(1-x)(1-y)(1-z)} \left(\sum_c t^c\right) \left(\sum_b (-1)^b e_b(B) t^b\right) \left(\sum_a w_a t^a\right) \\ &= \frac{1}{(1-x)(1-y)(1-z)} \left(\sum_i f_i t^i\right) \left(\sum_a w_a t^a\right) \\ &= \frac{1}{(1-x)(1-y)(1-z)} \sum_n F_n t^n, \end{aligned}$$

where

$$f_i := \sum_{d=0}^{i} (-1)^d e_d(E) \quad \text{and} \quad F_n = \sum_{i=0}^{n} f_i w_{n-i}.$$

As $f_4 = f_5 = f_6 = f_7 = \cdots$, we have

$$\begin{aligned} F_n &= f_0 w_n + f_1 w_{n-1} + f_2 w_{n-2} + f_3 w_{n-3} + f_4 w_{n-4} + f_4 w_{n-5} + f_4 w_{n-6} + \cdots \\ &= A(f_0(xy)^3 + f_1(xy)^2 + f_2(xy) + f_3)(xy)^{n-3} + A f_4 \sum_{i=0}^{n-4} (xy)^i + \text{cyclic}. \end{aligned}$$

In this expression, any term with a negative power of xy (or yz or zx) is interpreted as zero. Moreover, the word 'cyclic' means that one is to add the two terms with (A, B, C) and (x, y, z) cyclically permuted. A little algebra [14] reveals that

$$\frac{(f_0(xy)^3 + f_1(xy)^2 + f_2(xy) + f_3}{(1-x)(1-y)(1-z)} = x^2y^2 + xy + 1 - xyz$$

[14]...and a little help from MAPLE.

and

$$\frac{f_4}{(1-x)(1-y)(1-z)} = 1 - xyz.$$

Therefore, setting

$$F_n' := \frac{F_n}{(1-x)(1-y)(1-z)},$$

we find

$$\begin{aligned}
F_n' &= A(x^2y^2 + xy + 1 - xyz)(xy)^{n-3} + A(1-xyz)\sum_{i=0}^{n-4}(xy)^i + \text{cyclic} \\
&= A\sum_{i=0}^{n-1}(xy)^i - Axyz\sum_{i=0}^{n-3}(xy)^i + \text{cyclic} \\
&= \frac{1}{\Delta}(x-y)\left\{\sum_{i=1}^{n}(xy)^i - xyz\sum_{i=1}^{n-2}(xy)^i\right\} + \text{cyclic.}
\end{aligned}$$

Using Jacobi's alternant definition of the Schur functions gives

$$\begin{aligned}
&\frac{1}{\Delta}\left\{(x-y)(xy)^i + (y-z)(yz)^i + (z-x)(xz)^i\right\} \\
&\quad = \left(\frac{1}{xyz}\right)\left(\frac{x^{i+2}y^{i+1}z \pm \text{perms}}{\Delta}\right) \\
&\quad = \frac{1}{xyz}s_{(i,i,1)}(x,y,z) \\
&\quad = s_{(i-1,i-1,0)}(x,y,z).
\end{aligned}$$

Hence,

$$F_n' = \sum_{i=1}^{n} s_{(i-1,i-1,0)} - \sum_{i=1}^{n-2} s_{(i,i,1)}.$$

We can make this look a little nicer by observing that each term in $s_{(i-1,i-1,0)}$ has even total degree, and each term in $s_{(i,i,1)}$ has odd total degree, so we may drop all the minus signs and shift indices to obtain the main result.

Proposition 1. *For $0 \le k \le n-1$, let χ_k denote the character of the representation $\chi^{(n-k,1^k)}$, and set $g_{jk\ell} = \langle \chi_j\chi_k, \chi_\ell\rangle$. Then*

$$g_{jk\ell} = [x^j y^k z^\ell]\left(\sum_{i=0}^{n-1} s_{(i,i,0)} + \sum_{i=1}^{n-2} s_{(i,i,1)}\right). \tag{5}$$

From this result we see immediately that $g_{j,k,\ell} = 0$ if $j + k + \ell > 2n - 2$. But we may extract more detailed information from (5) by expanding the Schur functions in terms of monomials. Recall that, if λ is a partition of m and α is a weak composition of m, a *semi-standard Young tableau* (SSYT) of *shape* λ and *type* $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a tableau of shape λ which contains α_j j's, for $1 \le j \le m$, such that the rows are weakly increasing across and the columns are strictly increasing down. The combinatorial definition of Schur functions is

$$s_\lambda = \sum_\alpha K_{\lambda\alpha} x^\alpha, \tag{6}$$

summed over all weak compositions $\alpha \models m$, where $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ and the Kostka number $K_{\lambda\alpha}$ is the number of SSTY of shape λ and type α. As Schur functions are symmetric, $K_{\lambda\alpha}$ cannot depend on the order of the elements of α ([26], Proposition xxx), so one can also write

$$s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu, \tag{7}$$

where now the sum goes over all partitions $\mu \vdash m$, and m_μ is the monomial symmetric function.

It is not difficult to find the Kostka numbers appearing in (5). Consider first the Kostka numbers of the form $K_{(i,i,0),(j,k,\ell)}$. By virtue of the column constraint for semistandard Young tableaux, the only possible columns of a two-rowed tableau filled with the entries 1, 2, and 3 are of the form

$$P := \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, \quad Q := \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array}, \quad \text{and} \quad R := \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}.$$

Call these tableaux 'blocks'. By virtue of the row constraint, any particular tableau corresponds to a word of the form $P^p Q^q R^r$, where lower case letters represent the multiplicities of each letter. Any particular choice of word yields a unique tableau. In particular, $K_{(i,i,0),(j,k,\ell)} \in \{0, 1\}$. There is no restriction on the number of blocks of each kind beyond the fact that the total number must fit in the given diagram, so the number of tableaux of shape $(i, i, 0)$ is the number of solutions to $p + q + r = i$ in nonnegative integers. In particular, the number of solutions is just $\binom{i+2}{2}$. Moreover, we have

$$\begin{pmatrix} j \\ k \\ \ell \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ k \\ \ell \end{pmatrix}.$$

Hence, the conditions $p \geq 0$, $q \geq 0$, and $r \geq 0$ are equivalent to the constraints

$$\ell \leq j+k, \quad k \leq j+\ell, \quad \text{and} \quad j \leq k+\ell. \tag{8}$$

An almost identical situation holds for the Kostka numbers of the form $K_{(i,i,1),(j,k,\ell)}$, because the first column of any tableau of shape $(i,i,1)$ must contain the numbers 1, 2, and 3. If we remove this column, we are left with a two-rowed tableau of shape $(i-1,i-1,0)$. Again, we find that $K_{(i,i,1),(j,k,\ell)} \in \{0,1\}$. Moreover, the number of SSYT of shape $(i,i,1)$ and type (j,k,ℓ) is $\binom{i+1}{2}$. Constructing the tableaux with blocks as before, we find that

$$\begin{pmatrix} j-1 \\ k-1 \\ \ell-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} j-1 \\ k-1 \\ \ell-1 \end{pmatrix}.$$

Now the conditions $p \geq 0$, $q \geq 0$, and $r \geq 0$ are equivalent to the constraints

$$\ell \leq j+k-1, \quad k \leq j+\ell-1, \quad \text{and} \quad j \leq k+\ell-1. \tag{9}$$

Combining (8) and (9) yields the following. [15]

Proposition 2. *Let $((P)) = 1$ if the proposition P is true, and zero otherwise. If $j+k+\ell$ is even, then*

$$g_{ijk} = ((j \leq k+\ell))((k \leq j+\ell))((\ell \leq j+k)).$$

If $j+k+\ell$ is odd, then

$$g_{ijk} = ((j \leq k+\ell-1))((k \leq j+\ell-1))((\ell \leq j+k-1)).$$

We can also characterize the nonzero Kronecker coefficients geometrically. Let $\mathbb{N}$ denote the natural numbers (including zero). To any multivariate polynomial $f = \sum_{\alpha \in \mathbb{N}^m} c_\alpha x^\alpha$ we associate a convex polytope in $\mathbb{R}^m$, namely its *Newton polytope* $N(f)$, which is the convex hull in $\mathbb{R}^m$ of all the exponent vectors $\alpha \in \mathbb{N}^m$ corresponding to the nonzero coefficients c_α. The Newton polytope is *saturated* if $c_\alpha \neq 0$ whenever $\alpha \in N(f)$. [16] The study of saturated Newton polytopes in algebraic combinatorics has attracted much attention recently (see, *e.g.*, [15]).

[15] Compare with [25], Theorem 3(4).

[16] If $N(f)$ is saturated, every lattice point in $N(f)$ occurs as an exponent vector of f. Most polynomials fail this condition.

As shown in ([15], Proposition 2.5), Schur polynomials have saturated Newton polytopes. The proof is a consequence of the properties of Kronecker coefficients and a fundamental theorem of Rado on inequalities. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an integer vector. The convex hull of the $\mathfrak{S}_m$ orbit of λ, written $\mathcal{P}_\lambda$, is called a *generalized permutahedron.* [17] As is well known, the set of partitions of a fixed positive integer m is partially ordered by dominance, where $\mu = (\mu_1, \ldots, \mu_r) \vdash m$ is dominated by $\lambda = (\lambda_1, \ldots, \lambda_s) \vdash m$, written $\mu \prec \lambda$, provided

$$\sum_{i=1}^{h} \mu_i \le \sum_{i=1}^{h} \lambda_i \text{ for all } h \ge 1.$$

The following two results are fundamental.

Lemma 1 ([26], Proposition 7.10.5 and Ex. 7.12)**.**

$$K_{\lambda\mu} \neq 0 \Leftrightarrow \mu \prec \lambda.$$

Lemma 2 ([22], Theorem 1)**.**

$$\mathcal{P}_\mu \subseteq \mathcal{P}_\lambda \Leftrightarrow \mu \prec \lambda.$$

Using these lemmata, together with elementary properties of Newton polytopes shows that

$$N(s_\lambda(x_1, \ldots, x_m)) = \mathcal{P}_\lambda$$

is saturated. We therefore have the following result.

Proposition 3. *For* $0 \le j, k, \ell \le n-1$ *we have*

$$g_{jk\ell} = \begin{cases} 1, & \textit{if } (j,k,\ell) \in \mathcal{P}_{(i,i,0)} \textit{ for some } 0 \le i \le n-1, \\ 1, & \textit{if } (j,k,\ell) \in \mathcal{P}_{(i,i,1)} \textit{ for some } 1 \le i \le n-2, \textit{ and} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. By virtue of the preceding remarks, we need only show that the nonzero values of the Kronecker coefficients must equal unity. (Observe that the three cases above are disjoint, as the Schur functions $s_{(i,i,0)}$ and $s_{(i',i',1)}$ appearing in (5) are homogeneous of different degrees.) But this follows immediately from the discussion preceding Proposition 2. $\square$

[17] The ordinary permutahedron is $\mathcal{P}_{(1,2,\ldots,n)}$. For more about generalized permutahedra, see, *e.g.*, [3], [20] and [21].

Corollary 1. *Set $b := (j+k+\ell) \bmod 2$ and define $T_b(i)$ to be the triangle in $\mathbb{R}^3$ whose vertices are (i,i,b), (i,b,i), and (b,i,i). Then the nonzero (and hence unit) Kronecker coefficients $g_{jk\ell}$ are in one-one correspondence with the integer points in (and on) $\bigcup_{i\in I(b)} T_b(i)$, where $I(0) = \{0,\ldots,n-1\}$ and $I(1) = \{1,\ldots,n-2\}$.*

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