A major index determinant

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Abstract

We offer a simple proof of a determinant evaluation of Thibon.

1 Introduction

The evaluation of combinatorially defined determinants has been a favorite pastime of generations of mathematicians. Such evaluations are prized both for their intrinsic beauty as well as their utility, especially when they admit simple factorizations. We are interested here in one such determinant evaluation involving a certain permutation statistic.

Let $\mathfrak{S}_n$ denote the set of all permutations of $\{1, 2, \ldots, n\}$. We view permutations as bijections $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and write them in one-line notation as $\sigma = [\sigma(1), \sigma(2), \ldots, \sigma(n)]$ or $[\sigma_1, \sigma_2, \ldots, \sigma_n]$. We multiply permutations right to left, so that $(\sigma\tau)(i) = \sigma(\tau(i))$. The index i is a *descent* of σ if $\sigma(i) > \sigma(i + 1)$. (Similarly, i is an *ascent* of σ if $\sigma(i) < \sigma(i + 1)$.) The *descent set* of a permutation σ is, not surprisingly, the set of descents of σ:

$$\mathrm{Des}(\sigma) := \{j : 1 \le j < n \text{ and } \sigma(j) > \sigma(j + 1)\}$$

The *major index* of a permutation is the sum of its descents:

$$\operatorname{maj}\sigma := \sum_{j\in\operatorname{Des}(\sigma)} j.$$

Among the many striking determinant evaluations contained in the beautiful survey article of Christian Krattenthaler [8] is the following one:

$$\det_{\sigma,\tau\in\mathfrak{S}_n}\left(q^{\operatorname{maj}(\sigma\tau^{-1})}\right) = \prod_{k=2}^{n}(1-q^k)^{n!(k-1)/k}. \tag{1}$$

In ([8], Appendix C), Krattenthaler gives a proof of (1), due to Jean-Yves Thibon, using noncommutative symmetric functions. [1] The purpose of this work is to offer a simpler proof of (1) that avoids the machinery of noncommutative symmetric functions. Our approach, which uses only computations in the symmetric group algebra, is inspired by a remark of Manfred Schocker at the end of [9].

2 A symmetric group algebra identity

Define

$$\kappa_n(q) := \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{maj}(\sigma)}\sigma \in \mathbb{C}\mathfrak{S}_n,$$

where $\mathbb{C}\mathfrak{S}_n$ is the (complex) group algebra of $\mathfrak{S}_n$. [2] Let ρ be the (left) regular representation of $\mathfrak{S}_n$ (extended linearly to $\mathbb{C}\mathfrak{S}_n$). Then

$$\rho(\kappa_n(q))\rho(\tau) = \rho(\kappa_n(q)\tau) = \sum_{\pi\in\mathfrak{S}_n} q^{\operatorname{maj}(\pi)}\rho(\pi\tau) = \sum_{\sigma\in\mathfrak{S}_n} q^{\operatorname{maj}(\sigma\tau^{-1})}\rho(\sigma).$$

It follows that

$$\rho(\kappa_n(q))_{\sigma\tau} = q^{\operatorname{maj}(\sigma\tau^{-1})},$$

so that, to evaluate the left hand side of (1) we must evaluate $\det\rho(\kappa_n(q))$.

Let

$$\gamma_j := (j\ j-1\ \cdots\ 1)$$

[1] As shown in [4], the algebra of noncommutative functions is canonically isomorphic to Solomon's descent algebra [10].

[2] The notation κ_n is used in the literature to honor Alexander Klyachko, who first observed [6] that, if ζ is a primitive n^{th} root of unity, then $(1/n)\kappa_n(\zeta)$ is an idempotent of $\mathbb{C}\mathfrak{S}_n$. For a nice presentation of the constellation of ideas surrounding the Klyachko idempotent, see [3].

denote the reversed j-cycle. By convention, $\gamma_1 = 1$, where 1 is the identity element of $\mathfrak{S}_n$. In ([9], Section 3), Schocker states without proof the following theorem, which appears (also without proof) in a footnote of a paper by Dieter Blessenohl and Hartmut Laue ([2], Eq. 9). [3]

Theorem 1.

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} \sigma = (1 + q\gamma_n + q^2\gamma_n^2 + \cdots + q^{n-1}\gamma_n^{n-1}) \cdots (1 + q\gamma_3 + q^2\gamma_3^2)(1 + q\gamma_2)$$

As it is difficult to find a proof of Theorem 1 in the literature, and as the result is central to our discussion, we supply a proof in Section 3 below. [4] Before doing so, however, let us show how it easily implies Equation (1). The trick is to define (*cf.*, [9], Section 3)

$$\omega_n(q) := (1 - q\gamma_2)(1 - q\gamma_3) \cdots (1 - q\gamma_n),$$

and then observe that

$$\kappa_n(q)\omega_n(q) = (1 - q^2)(1 - q^3) \cdots (1 - q^n) \cdot 1.$$

Hence,

$$\rho(\kappa_n(q)) = \rho(\omega_n(q))^{-1} \prod_{k=2}^{n} (1 - q^k).$$

Taking determinants of both sides gives

$$\det \rho(\kappa_n(q)) = \det \rho(\omega_n(q))^{-1} \prod_{k=2}^{n} (1 - q^k)^{n!} = \prod_{k=2}^{n} \frac{(1 - q^k)^{n!}}{\det \rho(1 - q\gamma_k)}$$

(because the regular representation has dimension $n!$).

It remains to find $\det \rho(1 - q\gamma_k)$, the characteristic polynomial of the cycle γ_k in the regular representation. [5] The regular representation is just the permutation representation of $\mathfrak{S}_n$ acting on itself by left multiplication. Fix an element $\sigma \in \mathfrak{S}_n$. Its orbit under the

[3] We have reversed the order of multiplication to align the formula with our conventions.

[4] The result is implicit in [7]. See also [12].

[5] Several authors, including Phil Hanlon, Richard Stanley, and John Stembridge, have investigated the characteristic polynomials of the cycles γ_k in the irreducible representations of $\mathfrak{S}_n$, in connection with a determinant evaluation of Alexander Varchenko (see below). For a discussion and further references, see [5].

left action by γ_k has size k (namely, $\{\sigma, \gamma_k\sigma, \gamma_k^2\sigma, \dots, \gamma_k^{k-1}\sigma\}$). Hence, $\mathfrak{S}_n$ breaks up into $n!/k$ disjoint orbits under the action of γ_k. The characteristic polynomial of γ_k on a single orbit is just $1 - q^k$. Thus,

$$\det \rho(1 - q\gamma_k) = (1 - q^k)^{n!/k}.$$

We therefore conclude that

$$\det \rho(\kappa_n(q)) = \prod_{k=2}^{n} \frac{(1 - q^k)^{n!}}{(1 - q^k)^{n!/k}} = \prod_{k=2}^{n} (1 - q^k)^{n!(k-1)/k},$$

as desired.

3 A proof of Theorem 1

We will prove Theorem 1 in the following equivalent form. [6] Recall that a sequence $(i_1, i_2, i_3, \dots, i_n)$ is *subexcedant* if $0 \le i_j \le j - 1$. Denote by S_n the set of all subexcedant sequences of length n, and observe that $|S_n| = |\mathfrak{S}_n|$.

Theorem 2. *The following hold.*

1. *Every $\sigma \in \mathfrak{S}_n$ can be written* uniquely *as*

$$\sigma = \gamma_n^{i_n} \cdots \gamma_3^{i_3} \gamma_2^{i_2} \gamma_1^{i_1}, \tag{2}$$

where $(i_1, i_2, i_3, \dots, i_n) \in S_n$.

2. *If σ is written in the above form,*

$$\operatorname{maj} \sigma = i_1 + i_2 + \cdots + i_n. \tag{3}$$

Proof. 1. Let $\sigma \in \mathfrak{S}_n$ be given. We show that there exists a subexcedent sequence $(j_1, j_2, \dots, j_n) \in S_n$ such that

$$\gamma_1^{j_1} \gamma_2^{j_2} \gamma_3^{j_3} \cdots \gamma_n^{j_n} \sigma = 1. \tag{4}$$

We then obtain the result by setting $i_k := (k - j_k) \bmod k$.

[6] This form of the theorem appears (again without proof) in [1].

According to our conventions, left multiplication of σ by γ_k shifts the first k values of σ cyclically down by 1:

$$\gamma_k[\sigma_1, \sigma_2, \ldots, \sigma_n] = [(\sigma_1 - 1) \bmod k, (\sigma_2 - 1) \bmod k, \ldots, (\sigma_k - 1) \bmod k, \sigma_{k+1}, \ldots, \sigma_n]. \quad (5)$$

Set $\sigma' = \gamma_n^{j_n}\sigma$, where $j_n = \sigma_n \bmod n$. Then $\sigma'_n = n$, and the remaining entries of σ' are less than n. Next, set $\sigma'' = \gamma_{n-1}^{j_{n-1}}\sigma'$, where $j_{n-1} = \sigma'_{n-1} \bmod (n-1)$. Then $\sigma''_{n-1} = n-1$ (and $\sigma''_n = n$ still, because γ_{n-1} does not affect the n^{th} entry of σ'). Continuing in this way, we eventually reach the identity permutation. Moreover, $0 \le j_k \le k-1$ for all k.

2. Set $\sigma = \gamma_n^{\ell}\gamma_{n-1}^{i_{n-1}} \cdots \gamma_2^{i_2}\gamma_1^{i_1} \in \mathfrak{S}_n$. We proceed by induction on n and ℓ. The result is trivially true for $n = 1$ (and $\ell = 0$), so assume it holds for $n-1$ and $\ell < n-1$. Then it is enough to show that

$$\operatorname{maj}(\gamma_n\sigma) = \operatorname{maj}(\sigma) + 1. \quad (*)$$

Set $\sigma' := \gamma_n\sigma$. There are three cases.

i) $\sigma_1 = 1$. Then the first position of σ is always an ascent. Then $\sigma'_1 = n$, so the first position becomes a descent. Any other descents present in σ remain at the same positions in σ', because $\sigma_k > 1$ for $k > 1$, and thus, subtracting 1 from each entry lowers each entry by the same amount, whence $(*)$ holds.

ii) $\sigma_i = 1$ for $2 \le i \le n-1$. Then $i-1$ is a descent of σ and i is an ascent of σ, so $i-1$ is an ascent of σ' and i is a descent of σ'. As before, all the other descents remain unchanged. Again, $(*)$ obtains.

iii) $\sigma_n = 1$. This case is impossible, because $\ell < n-1$.

□

4 Remarks

If we apply the trivial representation to both sides of the expression appearing in Theorem 1, we obtain

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{maj}(\sigma)} = (1 + q + q^2 + \cdots + q^{n-1}) \cdots (1 + q + q^2)(1 + q).$$

Thus, the generating function on the right hand side counts the number of permutations by major index. As MacMahon first showed (see [11]), the same generating function also counts permutations by inversion number. (An *inversion* of a permutation σ is a pair (i,j) with $i < j$ and $\sigma(i) > \sigma(j)$. The *inversion number* $\mathrm{inv}(\sigma)$ is the number of inversions of σ.) That is, the inversion number and the major index are *equidistributed*. Because of this close connection between the two statistics, one might expect that there would be an analogue of (1) for the inversion number. Such an analogue was discovered independently by Alexander Varchenko [13] and Don Zagier [14]. [7]

$$\det_{\sigma,\tau\in\mathfrak{S}_n}\left(q^{\mathrm{inv}(\sigma\tau^{-1})}\right) = \prod_{k=2}^{n}(1-q^{k(k-1)})^{\binom{n}{k}(k-2)!(n-k+1)!}. \tag{6}$$

Zagier's proof of (6) (unlike that of Varchenko, which is more general but more complicated) uses a clever recursive approach in the symmetric group algebra. Is there a simpler approach to (6) similar to the one given above?

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