On the spectrum of the perfect matching derangement graph

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Abstract

We prove the alternating sign conjecture for the perfect matching derangement graph.

1 Introduction

A *matching* in a graph is a collection of pairwise disjoint edges. A matching is *perfect* if it meets each vertex of the graph precisely once. Let $\mathcal{M}_{2n}$ denote the set of all perfect matchings of the complete graph K_{2n}. We construct a graph Γ_{2n} on $\mathcal{M}_{2n}$, called the *perfect matching derangement graph*, as follows: the vertices of Γ_{2n} are the elements of $\mathcal{M}_{2n}$, and M and M' are adjacent in Γ_{2n} if $M \cap M' = \varnothing$. [1] It is not difficult to see that

[1]The use of the word 'derangement' in this context is somewhat awkward, but now standard. One can draw an analogy between the graph Γ_{2n} and the *derangement graph*, whose vertices are the set of permutations on an n element set, and whose edges are pairs $\{\pi, \sigma\}$ of permutations that are derangements of one another, meaning that, in one-line notation, π and σ disagree in every position. In that sense, one could say that π and σ do not intersect. Similarly, if two perfect matchings M and M' do not intersect, it is said that M and M' are derangements of one another.

Γ_{2n} has

$$(2n-1)!! = (2n-1)(2n-3)\cdots 1 \tag{1}$$

vertices.

The perfect matching derangement graph Γ_{2n} has attracted much attention recently [2], as part of a larger study of Erdős-Ko-Rado type theorems. [3] A family $\mathcal{F} \subseteq \mathcal{M}_{2n}$ of matchings is *intersecting* if $M \cap M' \neq \varnothing$ for every pair $\{M, M'\} \in \mathcal{F}$. It is *trivially intersecting* if it consists of all the perfect matchings containing a fixed edge of K_{2n}. Another simple counting exercise shows that a trivially intersecting family contains $(2n-3)!!$ elements. Using a counting argument in the more general context of uniform partitions, Meagher and Moura [11] proved the following Erdős-Ko-Rado type theorem for perfect matchings.

Theorem 1. *Every intersecting family of perfect matchings has at most* $(2n-3)!!$ *elements. Any family meeting this bound is trivially intersecting.*

As an intersecting family of $\mathcal{M}_{2n}$ is just an independent set of Γ_{2n}, the first part of the theorem can be restated as follows.

Theorem 2. *The independence number of* Γ_{2n} *is* $(2n-3)!!$.

Theorem 2 was obtained again later (and independently) by Godsil and Meagher [2] and Lindzey [9] using algebraic methods such as the Delsarte-Hoffman bound, which relates the independence number of a graph to its least eigenvalue. This approach leads naturally to a study of the spectrum of Γ_{2n}. In the course of their investigations, Lindzey conjectured [9], and Godsil and Meagher [2] proved, the following result, which implies Theorem 2.

Theorem 3. *The least eigenvalue of the perfect matching derangement graph* Γ_{2n} *is*

$$-\frac{d_{2n}}{2n-2},$$

where d_{2n} *is the degree of* Γ_{2n}.

[2] See, *e.g.*, [7], and references therein, as well as the bibliography below.

[3] For a beautiful introduction to this constellation of ideas, see [1].

Remark 4. *The degree d_{2n} equals the number of perfect matchings that are disjoint from a given perfect matching. By an inclusion-exclusion argument we find*

$$d_{2n} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2(n-i)-1)!!. \tag{2}$$

Further study of the eigenvalues of Γ_{2n} reveal an interesting pattern. Just as the eigenvalues of the (ordinary) derangement graph are indexed by integer partitions, so too are the eigenvalues of the perfect matching derangement graph. Based on a result of [14], it was first observed and proved by Ku and Wales [5] (and later more simply by Ku and Wong [6]) that the eigenvalues η_λ of the derangement graph possess the so-called *alternating sign property*, namely that $\mathrm{sgn}(\eta_\lambda) = (-1)^{|\lambda|-\lambda_1}$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Examination of the eigenvalues of Γ_{2n} led Lindzey [9] and Godsil and Meagher [1] to conjecture that a similar result holds for Γ_{2n}.

Conjecture 5. *The matching derangement graph Γ_{2n} satisfies the alternating sign property. That is, for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$,*

$$\mathrm{sgn}(\eta_\lambda) = (-1)^{|\lambda|-\lambda_1}.$$

Recently, Ku and Wong [7] proved that this conjecture holds for partitions of the form $\lambda = 1^n$ and $\lambda = 2^{2m}1^{n-2m}$. The purpose of this work is to prove the conjecture in general. To do so, we employ the shifted Jack functions introduced by Knop and Sahi [4] and developed by Okounkov and Olshanski [13].

2 The eigenvalues of Γ_{2n}

As observed by Godsil and Meagher [1] and others, the perfect matching derangement graph is a union of graphs of an association scheme. This leads to a formula for the eigenvalues η_λ of Γ_{2n}. (For more details, see [7], whose treatment we follow here. See also [10] (especially Section VII.2) and [16] for proofs of some of the assertions below, as well as for additional background information.) Let $\mathfrak{S}_{2n}$ denote the symmetric group on $2n$ elements. This group acts naturally on perfect matchings. Label the vertices of

K_{2n} by $\{1,\ldots,2n\}$, and write $M = \{i_1,j_1\}\cdots\{i_n,j_n\}$ for the matching whose edges are $\{i_k,j_k\}_{k=1}^n$. Then gM is the matching $\{g(i_1),g(j_1)\}\cdots\{g(i_n),g(j_n)\}$. The stabilizer of a given matching is the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_n$, which is isomorphic to H_n, the hyperoctahedral group of order $|H_n| = 2^n n!$. We may therefore identify the distinct perfect matchings with the left cosets $\mathfrak{S}_{2n}/H_n$.

To every $g \in \mathfrak{S}_{2n}$ we can associate its *coset type*, as follows. Let $M \in \mathcal{M}_{2n}$. Now superimpose the two perfect matchings M and gM to form a graph Δ. Every vertex of the Δ has degree two, so its is a union of cycles. Every cycle of Δ has even length, because successive edges alternate between matchings. Let ν be the partition of n obtained by halving the cycle lengths of Δ and writing them in nonincreasing order. The partition ν is the coset type of g. It is independent of the particular matching. Indeed, it can be shown ([10], p. 401) that g and g' have the same coset type if and only if g' lies in the double coset $H_n g H_n$, whence the nomenclature.

As g and g^{-1} have the same coset type, $(\mathfrak{S}_{2n}, H_n)$ constitutes a Gel'fand pair, with associated zonal spherical functions

$$\omega^\lambda(g) = \frac{1}{|H_n|}\sum_{h\in H_n}\chi^{2\lambda}(gh), \tag{3}$$

where $\chi^{2\lambda}$ is the irreducible character of $\mathfrak{S}_{2n}$ indexed by $2\lambda \vdash 2n$. It can be shown that $\omega^\lambda(g)$ depends only on the coset type ν of g, and that the eigenvalues of Γ_{2n} can be written ([7], p. 634)

$$\eta_\lambda = \sum_{\nu\in D_n} k_\nu \omega^\lambda(\nu), \tag{4}$$

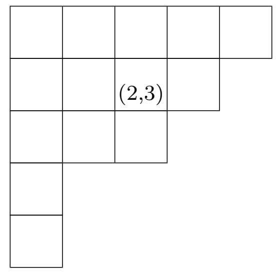
where D_n is the set of all partitions of n having no part of size 1, and

$$k_\nu = \frac{|H_n|}{z_{2\nu}} = \frac{|H_n|}{2^{\ell(\nu)}z_\nu}, \tag{5}$$

where $z_\nu = \prod_{i=1}^r m_i! i^{m_i}$ is the order of the centralizer of a permutation with cycle type $\nu = 1^{m_1}2^{m_2}\cdots r^{m_r}$, and $\ell(\nu) = \sum_i m_i$ is the length of the partition ν.

We take (4) as our starting point, and develop a formula for the eigenvalues of Γ_{2n}, from which the conjecture will follow immediately. To this end we employ what we call *shifted zonal polynomials*, which are special cases of the *shifted Jack polynomials* introduced by Knop and Sahi [4] and developed by Okounkov and Olshanski [13].

Before beginning, we need a few more definitions. We identify partitions with their Ferrers diagrams written in English style, so that the position of a cell of the partition is specified by the pair (i, j), where i denotes the row (counting from top to bottom) and j the column (counting from left to right), as illustrated below.



The *arm length* $a(\square)$, the *leg length* $\ell(\square)$, the *arm co-length* $a'(\square)$, and the *leg co-length* $\ell'(\square)$ are, respectively, the number of boxes to the right of, below, to the left of, and above, the box $\square$ (not including the box itself). If $\square$ is located at (i, j) in the partition $\lambda = (\lambda_1, \lambda_2, \dots)$,

$$a(\square) := \lambda_i - j \qquad\qquad a'(\square) := j - 1$$
$$\ell(\square) := \lambda'_j - i \qquad\qquad \ell'(\square) := i - 1,$$

where $\lambda'_j = |\{i : \lambda_i \geq j\}|$. The *hook length* $h(\square)$ at $\square \in \lambda$ is defined as

$$h(\square) := a(\square) + \ell(\square) + 1.$$

The *hook length product* $h(\lambda)$ is

$$h(\lambda) := \prod_{\square \in \lambda} h(\square).$$

The first step is to relate the zonal spherical functions ω^λ to the ordinary zonal polynomials Z_λ. According to equation (VII.2.13) on p. 405 of [10],

$$Z_\lambda = \sum_{\nu \vdash n} \frac{|H_n|}{z_{2\nu}} \omega^\lambda(\nu) p_\nu, \tag{6}$$

where p_ν is the power sum symmetric function of type ν. Comparing (4) and (6) we see that the eigenvalues of Γ_{2n} may be written

$$\eta_\lambda = Z_\lambda|_{p_1=0, p_2=p_3=\cdots=1}. \tag{7}$$

This expression is analogous to one employed in [14] to compute the eigenvalues of the ordinary derangement graph (and which is, in turn, based on the results of an exercise in [17]). There, the formula led to shifted complete functions (special cases of shifted Schur functions). Here, it will lead to shifted zonal functions.

Next, we write down an analogue of the Cauchy formula for the zonal polynomials ([10], p. 406)

$$\sum_\lambda h(2\lambda)^{-1} Z_\lambda(x) Z_\lambda(y) = \prod_{ij} (1 - x_i y_j)^{-1/2}, \tag{8}$$

where $h(2\lambda)$ is the product of the hook lengths of the partition 2λ. For later use we record the following formula for $h(2\lambda)$ ([10], p. 407):

$$h(2\lambda) = h_1(2\lambda) h_2(2\lambda), \tag{9}$$

where

$$h_i(2\lambda) = \prod_{\square \in \lambda} (2a(\square) + \ell(\square) + i), \qquad (i = 1, 2).$$

In particular,

$$h_1(2(k)) = \prod_{r=1}^{k} (2(k - r) + 1) = (2k - 1)!!. \tag{10}$$

Expanding the right hand side of (8) we have ([10], p. 407)

$$\sum_\lambda h(2\lambda)^{-1} Z_\lambda(x) Z_\lambda(y) = \prod_{r \geq 1} \exp\left\{ \frac{p_r(x) p_r(y)}{2r} \right\}.$$

Setting $p_1(y) = 0$ and $p_2(y) = p_3(y) = \cdots = 1$ gives

$$\sum_\lambda \frac{\eta_\lambda}{h(2\lambda)} Z_\lambda(x) = \prod_{r \geq 2} \exp\left\{ \frac{p_r(x)}{2r} \right\} = \exp\left\{ \sum_{r \geq 2} \frac{p_r(x)}{2r} \right\}.$$

By ([17], Equation (5.36), p. 21)

$$\sum_{k \geq 0} h_k t^k = \exp\left\{ \sum_{r \geq 1} \frac{p_r(x)}{r} t^r \right\}.$$

Setting $t = 1$ yields

$$\exp\left\{ \sum_{r \geq 2} \frac{1}{r} p_r(x) \right\} = e^{-h_1} \sum_{k \geq 0} h_k.$$

Hence,

$$\sum_\lambda \frac{\eta_\lambda}{h(2\lambda)} Z_\lambda(x) = e^{-h_1/2} \left(\sum_{k \geq 0} h_k \right)^{1/2}. \tag{11}$$

Recalling that the generating function for the homogenous symmetric polynomials is

$$\sum_k h_k t^k = \prod_k \frac{1}{1 - x_k t},$$

we may write (11) as

$$\sum_\lambda \frac{\eta_\lambda}{h(2\lambda)} Z_\lambda(x) = e^{-h_1/2} \prod_k \frac{1}{\sqrt{1 - x_k}}. \tag{12}$$

But, according to ([10], p. 407),

$$\prod_k \frac{1}{\sqrt{1 - x_k}} = \sum_k \frac{Z_k}{|H_k|},$$

where we write Z_k for $Z_{(k)}$. Hence

$$\sum_\lambda \frac{\eta_\lambda}{h(2\lambda)} Z_\lambda(x) = e^{-Z_1/2} \sum_k \frac{Z_k}{|H_k|} = \sum_{j,k} \frac{1}{|H_j||H_k|} (-1)^j Z_1^j Z_k, \tag{13}$$

where we used the fact that

$$Z_1 = h_1 = \sum_i x_i.$$

At this point, we wish to invoke some results of Okounkov and Olshanski [13] (which are based in part on earlier work of Knop and Sahi [4]). However, we encounter a small problem of normalizations. All the above formulae use the normalization of zonal polynomials (originally used by James [3], and subsequently by Macdonald [10] and Stanley [16]) for which the coefficient of $x_1 x_2 \cdots x_n$ is $n!$. But in [13], Okounkov and Olshanski normalize their zonal polynomials [4] (denoted there by $P_\mu(x; \frac{1}{2})$, and which we write as $P_\mu(x)$) so that $P_\lambda(x) = x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ + lower degree terms. Because of this, it is awkward to compare the two normalizations directly. Instead, we will obtain their relationship by comparing two Pieri-type formulae.

In ([13], Eq. (5.1) with $\theta = 1/\alpha = 1/2$), Okounkov and Olshanski associate a parameter $d^{\lambda/\mu}$ (there called the 'θ dimension' of the skew diagram λ/μ) by means of the Pieri-type formula

$$(\sum_i x_i)^j P_\mu(x) = \sum_{|\lambda|=|\mu|+j} d^{\lambda/\mu} P_\lambda(x). \tag{14}$$

[4]Actually, Okounkov and Olshanski discuss only Jack polynomials, which they write as $P_\mu(x;\theta)$. Here $\theta = 1/\alpha$, where α is the usual Jack parameter. But Macdonald and Stanley show (see, *e.g.*, [16], Proposition 1.2 or [10], p. 408) that the zonal polynomials are obtained from the Jack polynomials by setting $\alpha = 2$. For this reason, we call $P_\mu(x; \frac{1}{2})$ zonal polynomials.

They also show ([13], equation following (5.1)) that

$$d^{\lambda} := d^{\lambda/\varnothing} = \frac{|\lambda|!}{H(\lambda)}, \tag{15}$$

where

$$H(\lambda) = \prod_{\square \in \lambda} (a(\square) + \ell(\square)/2 + 1) = 2^{-|\lambda|} h_2(2\lambda). \tag{16}$$

Taking $\mu = \varnothing$ in (14) and using (15) and (16) we get

$$Z_1^j = \sum_{\lambda \vdash j} 2^{|\lambda|} \frac{|\lambda|!}{h_2(2\lambda)} P_{\lambda}(x). \tag{17}$$

Comparing this to the corresponding result of Stanley ([16], Proposition 2.3 with $\alpha = 2$), namely

$$Z_1^j = |H_j| \sum_{\lambda \vdash j} \frac{Z_{\lambda}}{h(2\lambda)}. \tag{18}$$

we conclude (using (9) and the fact that the zonal polynomials are linearly independent) that

$$P_{\lambda} = \frac{1}{h_1(2\lambda)} Z_{\lambda}. \tag{19}$$

Thus, Equation (14) can be written

$$Z_1^j Z_{\mu}(x) = \sum_{|\lambda| = |\mu| + j} d^{\lambda/\mu} \left(\frac{h_1(2\mu)}{h_1(2\lambda)} \right) Z_{\lambda}(x). \tag{20}$$

From (20), we obtain

$$Z_1^j Z_k = \sum_{|\lambda| = j + k} d^{\lambda/k} \left(\frac{h_1(2k)}{h_1(2\lambda)} \right) Z_{\lambda}.$$

Plugging this into (13) gives

$$\sum_{\lambda} \frac{\eta_{\lambda}}{h(2\lambda)} Z_{\lambda}(x) = \sum_{j,k,|\lambda| = j+k} \frac{(-1)^j h_1(2k)}{|H_j||H_k| h_1(2\lambda)} d^{\lambda/k} Z_{\lambda}$$

and thus (equating coefficients of Z_{λ})

$$\frac{\eta_{\lambda}}{h(2\lambda)} = \sum_{j+k=|\lambda|} \frac{(-1)^j h_1(2k)}{|H_j||H_k| h_1(2\lambda)} d^{\lambda/k} = \sum_{k=0}^{|\lambda|} \frac{(-1)^{|\lambda|-k} h_1(2k)}{|H_{|\lambda|-k}||H_k| h_1(2\lambda)} d^{\lambda/k}. \tag{21}$$

Now we appeal to Proposition (5.2) in [13], which states that

$$\frac{d^{\lambda/\mu}}{d^{\lambda}} = \frac{P^*_{\mu}(\lambda)}{|\lambda|(|\lambda|-1)\cdots(|\lambda|-|\mu|+1)}, \tag{22}$$

where $P^*_{\mu}(x)$ is the $\theta = 1/2$ case of the shifted Jack polynomials of [4] and [13]. (See below.) We call $P^*_{\mu}(x)$ a *shifted zonal polynomial.* Plugging this into (21) gives (using (9), (15), and (16))

$$\begin{aligned}
\eta_\lambda &= \sum_{k=0}^{|\lambda|} \frac{(-1)^{|\lambda|-k} h_1(2k) h(2\lambda) d^{\lambda}}{|H_{|\lambda|-k}||H_k| h_1(2\lambda)} \left(\frac{P^*_k(\lambda)}{(|\lambda|)(|\lambda|-1)\cdots(|\lambda|-k+1)} \right) \\
&= \sum_{k=0}^{|\lambda|} \frac{(-1)^{|\lambda|-k} h_1(2k) 2^{|\lambda|} |\lambda|!}{|H_{|\lambda|-k}||H_k|} \left(\frac{P^*_k(\lambda)}{(|\lambda|)(|\lambda|-1)\cdots(|\lambda|-k+1)} \right) \\
&= \sum_{k=0}^{|\lambda|} (-1)^{|\lambda|-k} h_1(2k) \frac{|\lambda|!}{(|\lambda|-k)!k!} \left(\frac{P^*_k(\lambda)}{(|\lambda|)(|\lambda|-1)\cdots(|\lambda|-k+1)} \right) \\
&= \sum_{k=0}^{|\lambda|} \frac{(-1)^{|\lambda|-k}(2k-1)!!}{k!} P^*_k(\lambda).
\end{aligned}$$

To make this look a bit nicer, we define a renormalized version of the shifted zonal polynomials, namely

$$F^*_k(x) := \frac{(2k-1)!!}{k!} P^*_k(x). \tag{23}$$

For want of a better term, we call these *F-polynomials.* With this definition, we may summarize our results as follows.

Theorem 6. *The eigenvalues of the perfect matching derangement graph Γ_{2n} are labeled by partitions $\lambda \vdash n$, and are given by*

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} F^*_k(\lambda).$$

Remark 7. *The multiplicity of η_λ is $\dim \chi^{2\lambda}$ (see, e.g., [9], Theorem 4.3), but we do not need this here.*

Of course, this formula is not too helpful unless we can say something useful about the shifted zonal polynomials. As observed by Sahi [15] and Knop and Sahi [4] (see also

[13], Section 2), the shifted zonal polynomials $P^*_\mu(x)$ are the unique shifted symmetric polynomials (meaning that they are symmetric in the variables $z_i = x_i - i/2$, where $i = 1, \ldots, n$) with $\deg P^*_\mu \leq |\mu|$ satisfying

$$P^*_\mu(\lambda) = \begin{cases} H(\mu), & \text{if } \lambda = \mu, \text{ and} \\ 0, & \text{if } |\lambda| \leq |\mu|,\ \mu \neq \lambda. \end{cases} \tag{24}$$

Knop and Sahi [4] also showed that the shifted zonal polynomials satisfy a *vanishing property*

$$P^*_\mu(\lambda) = 0 \quad \text{unless } \mu \subseteq \lambda, \tag{25}$$

and that

$$P^*_\mu(x) = P_\mu(x) + \text{terms of lower degree.}$$

Okounkov and Olshanski [13] observed that the shifted zonal polynomials satisfy a *stabilization property*:

$$P^*_\mu(x_1, \ldots, x_n, 0) = P^*_\mu(x_1, \ldots, x_n).$$

Unfortunately, the description given above of the shifted zonal polynomials is too indirect to be of much use in the present context. In [13], Okounkov and Olshanski derive an explicit combinatorial formula for the shifted zonal polynomials (Eq. 2.4) which could potentially be applied to our problem. But rather than use that formula, we found it simpler to exploit a recurrence relation first obtained by Okounkov in the context of Macdonald polynomials (Eq. (7.4) in [12], adapted to our situation):

$$P^*_\mu(x_1, x_2, \ldots, x_n) = \sum_{\nu \prec \mu} \psi_{\mu/\nu} \langle x_1 \rangle_{\mu/\nu} P^*_\nu(x_2, x_3, \ldots, x_n), \tag{26}$$

where $P^*_\varnothing(x_1, \ldots, x_n) = 1$. In (26) we have [5]

$$\langle z \rangle_{\mu/\nu} := \prod_{\square \in \mu/\nu} (z - a'(\square) + \ell'(\square)/2) \tag{27}$$

and

$$\psi_{\mu/\nu} := \prod_{\square \in R_{\mu/\nu} - C_{\mu/\nu}} \frac{b_\nu(\square)}{b_\mu(\square)},$$

where

$$b_\lambda(\square) := \frac{2a_\lambda(\square) + \ell_\lambda(\square) + 1}{2a_\lambda(\square) + \ell_\lambda(\square) + 2},$$

[5] Note that $a'(\square)$ and $\ell'(\square)$ are computed relative to the origin of μ. Note also that, to be consistent with the initial condition, we must set $\langle z \rangle_\varnothing = \psi_\varnothing = 1$.

and $C_{\mu/\nu}$ (resp. $R_{\mu/\nu}$) is the union of the columns (resp. rows) of μ that meet μ/ν. The notation '$\nu \prec \mu$' means

$$\mu_1 \geq \nu_1 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_n.$$

Setting $\mu = (k)$ and $\nu = (j)$ in (26) gives

$$P_k^*(x_1, \ldots, x_n) = \sum_{j=0}^{k} \psi_{k/j} \langle x_1 \rangle_{k/j} P_j^*(x_2, \ldots, x_n). \tag{28}$$

By (27),

$$\langle z \rangle_{k/j} = \begin{cases} \prod_{r=j}^{k-1}(z-r) = (z-j)(z-j-1)\cdots(z-k+1), & \text{if } j < k, \\ 1, & \text{if } j = k, \text{ and} \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Also, as $a_\ell(r) = \ell - r$, we have

$$\begin{aligned} \psi_{k/j} &= \prod_{r=1}^{j} \frac{b_j(r)}{b_k(r)} \\ &= \prod_{r=1}^{j} \left(\frac{2a_j(r)+1}{2a_j(r)+2}\right)\left(\frac{2a_k(r)+2}{2a_k(r)+1}\right) \\ &= \prod_{r=1}^{j} \left(\frac{2j-2r+1}{2j-2r+2}\right)\left(\frac{2k-2r+2}{2k-2r+1}\right) \\ &= \prod_{r=1}^{j} \left(\frac{k-r+1}{j-r+1}\right)\left(\frac{2j-2r+1}{2k-2r+1}\right) \\ &= \binom{k}{j} \frac{(2j-1)!!(2k-2j-1)!!}{(2k-1)!!}. \end{aligned}$$

Plugging this into (28) gives

$$P_k^*(x_1, \ldots, x_n) = \sum_{j=0}^{k} \binom{k}{j} \frac{(2j-1)!!(2k-2j-1)!!}{(2k-1)!!} \langle x_1 \rangle_{k/j} P_j^*(x_2, \ldots, x_n).$$

Using (23) we get

$$F_k^*(x_1, \ldots, x_n) = \sum_{j=0}^{k} \frac{(2k-2j-1)!!}{(k-j)!} \langle x_1 \rangle_{k/j} F_j^*(x_2, \ldots, x_n).$$

Now define

$$[z]_{k/j} := \frac{(2(k-j)-1)!!}{(k-j)!}\langle z\rangle_{k/j}.$$

Then we get the following.

Theorem 8. *The F-polynomials obey the recurrence relation*

$$F_k^*(x_1,\dots,x_n) = \sum_{j=0}^{k}[x_1]_{k/j}F_j^*(x_2,\dots,x_n).$$

with $F_0^*(x_1,\dots,x_n) = 1$.

Running out the recurrence yields another expression for the F-polynomials.

Corollary 9. *The F-polynomials are given by*

$$F_k^*(x_1,\dots,x_n) = \sum_{k\geq j_1\geq j_2\geq\cdots\geq j_{n-1}\geq j_n=0}[x_1]_{k/j_1}[x_2]_{j_1/j_2}\cdots[x_n]_{j_{n-1}/j_n}.$$

We also get a nonnegativity result.

Corollary 10. *For any partition λ, $F_k^*(\lambda)\geq 0$.*

Proof. By induction on the length $\ell(\lambda)$ of λ. The case $\ell = 0$ is trivial, so assume that it holds for $\ell - 1$. By the vanishing condition (25) we have $F_j^*(\lambda_2,\lambda_3,\dots) = 0$ unless $\lambda_2 \geq j$. But our partitions are written in nonincreasing order, so $\lambda_1\geq\lambda_2$, which implies that $[\lambda_1]_{k/j}\geq 0$. The claim now follows from Theorem 8. □

Next we would like to combine Theorems 6 and 8. Before doing so, we observe that, by (29), $[z]_{k/j}$ vanishes for $j > k$. Also, by (25), $F_k^*(\lambda) = 0$ if $k > |\lambda|$ (indeed, for $k > \lambda_1$). So, we may safely ignore the upper limits on the sums appearing in both theorems and write

$$\eta_\lambda = (-1)^{|\lambda|}\sum_k(-1)^kF_k^*(\lambda_1,\dots,\lambda_n) = (-1)^{|\lambda|}\sum_k(-1)^k\sum_j[\lambda_1]_{k/j}F_j^*(\lambda_2,\dots,\lambda_n).$$

Interchanging the sums gives

$$\eta_\lambda = (-1)^{|\lambda|}\sum_j F_j^*(\lambda_2,\dots,\lambda_n)\sum_k(-1)^k[\lambda_1]_{k/j}.$$

Setting $\ell := k - j$ we have

$$\begin{aligned}\sum_k (-1)^k [x]_{k/j} &= \sum_{k \geq j} (-1)^k (2k-2j-1)!! \binom{x-j}{k-j} \\ &= (-1)^j \sum_{\ell \geq 0} (-1)^\ell (2\ell - 1)!! \binom{x-j}{\ell} \\ &= (-1)^x d_{2(x-j)}.\end{aligned}$$

from (2). Hence we obtain the following.

Theorem 11. *The eigenvalues of* Γ_{2n} *are given by*

$$\eta_\lambda = (-1)^{|\lambda| - \lambda_1} \sum_{j=0}^{\lambda_1} d_{2(\lambda_1 - j)} F_j^*(\lambda_2, \ldots, \lambda_n).$$

As both d_{2n} and $F_j^*(\lambda_2, \ldots, \lambda_n)$ are nonnegative, we finally obtain the desired conclusion.

Corollary 12. *Conjecture 5 is true.*

Note Added

After this article was submitted, C.-Y. Ku informed the author that he and his colleagues also proved Conjecture 5 at roughly the same time using different methods [8].

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