A Combinatorial Proof of a Symmetric Group Character Involution

Paul Renteln

Abstract. We give a short combinatorial proof based on the Murnaghan-Nakayama rule of the symmetric group character identity $\chi^\lambda \chi^{(1^n)} = \chi^{\lambda'}$, where λ' is the conjugate of the partition λ.

1. INTRODUCTION. Let $\mathfrak{S}_n$ be the symmetric group on n elements. If $\lambda \vdash n$ is a partition of n, we write either $\lambda = (\lambda_1 \ \ldots, \lambda_\ell)$, in which the parts are nonincreasing, or $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$, where m_i is the multiplicity of i in λ. We also represent partitions using their Ferrers diagrams, written in English style. The boxes of the diagram are labeled by pairs (i, j), where i and j increase down and to the right, respectively. The conjugate partition λ' is obtained by transposing the diagram about the diagonal. (See Figure 1.)

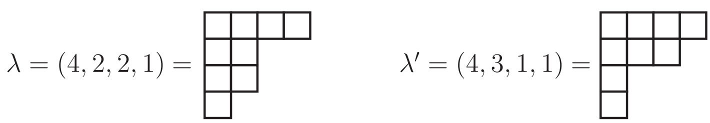

Figure 1. A partition and its conjugate.

As is well known, the irreducible characters χ^λ of $\mathfrak{S}_n$ are indexed by partitions $\lambda \vdash n$. There are two distinguished linear characters, namely the trivial character $\chi^n(\pi) = 1$, and the sign character $\chi^{(1^n)}(\pi) = (-1)^\pi$. The notation $(-1)^\pi$ denotes the sign of the permutation π, namely the image of π under the unique homomorphism $\mathfrak{S}_n \to \mathbb{Z}/2\mathbb{Z}$ that maps every transposition to -1.

It is a classical fact that twisting any irreducible character χ^λ by the sign character yields the irreducible character $\chi^{\lambda'}$:

$$\chi^\lambda \chi^{1^n} = \chi^{\lambda'}. \tag{1}$$

There are several proofs of (1). Macdonald [**4**, I.7.Ex.2] and Stanley [**5**, Ex. 7.78] both use the involution ω on symmetric functions that maps the Schur function s_λ to the Schur function $s_{\lambda'}$. Goldschmidt [**2**, Eq. 7.4] uses the fact that χ^λ is the unique irreducible component common to both $1_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$, and $(-1)_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n}$, where $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$ is the Young subgroup corresponding to λ, 1 and -1 denote the trivial and sign characters, respectively, and ψ_H^G is the induction of ψ from H up to G. A very short proof based on Young symmetrizers is given online at MathStackExchange [**3**]. Here, we offer a simple combinatorial proof based on the Murnaghan-Nakayama rule. [1]

doi.org/10.1080/00029890.2023.2242042
MSC: 05E10, 05E16

[1] After this note was submitted the author was made aware of a recent article by Chow and Paulhus [**1**], in which they also derive (1) from the Murnaghan-Nakayama rule, but with an involution argument (Lemma 17, attributed to Richard Stanley).

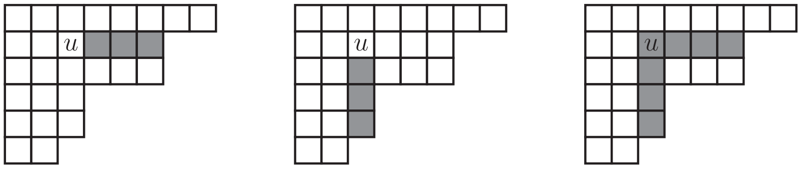

Figure 2. The arm, leg, and hook of u.

2. THE MURNAGHAN-NAKAYAMA RULE. We begin with some necessary definitions. Let λ be a partition. If u is a box of λ, the arm $a(u)$ (respectively, leg $\ell(u)$) of u is the set of all boxes to the right of u (respectively, below u). The hook $h(u)$ of u is the union of u and the arm and leg of u. (See Figure 2.) If u is located at (i, j), we have

$$|a(u)| = \lambda_i - j$$
$$|\ell(u)| = \lambda'_j - i$$
$$|h(u)| = \lambda_i + \lambda'_j - i - j + 1.$$

For a given $u \in \lambda$, the *border strip with hook point u* is obtained by following a contiguous path in λ, starting with the box at the end of the arm of λ, and continuing to the box at the end of the leg of λ, while remaining on the outside of the diagram. (See Figure 3.) Let B be a border strip with hook point $u = (i, j)$. Then it is easy to see that the size of B is $|h(u)|$. By definition, the *height* $\mathrm{ht}(B)$ of B is $|\ell(u)|$, namely one less than the number of rows of B.

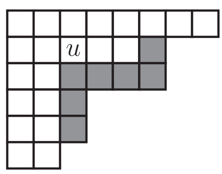

Figure 3. The border strip with hook point u.

Let $\lambda \vdash n$ be a partition. Form a sequence $T := (\lambda^1(= \lambda), \lambda^2, \lambda^3, \ldots, \lambda^{m+1}(= \emptyset))$ of partitions, where λ^{i+1} is obtained from λ^i by removing a border strip. The border strips themselves are the differences of the partitions, so we write $B_i = \lambda^i - \lambda^{i+1}$. Set $\alpha_i := |B_{m-i+1}|$. Then the sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a composition of n. The sequence T is called a *border strip tableau* of *shape* λ and *type* α. [2] The *height* of T is the sum of the heights of all the border strips within T:

$$\mathrm{ht}(T) = \sum_i \mathrm{ht}(B_i).$$

(See Figure 4.)

[2] More generally, one could allow some of the border strips in the sequence T to be empty, in which case α would be a weak composition of n.

4	3	2	1	1	1
3	3	2	1		
3	2	2	1		
3					

Figure 4. A border strip tableau of shape $(6, 4, 4, 1)$, type $(1, 5, 4, 5)$, and height $2 + 2 + 3 + 0 = 7$. The border strips are labeled in order of removal.

Every permutation in $\mathfrak{S}_n$ can be written as a product of disjoint cycles whose lengths sum to n. For instance, if $\pi = [\pi(1)\pi(2)\dots\pi(n)] = [7361524]$ then $\pi = (174)(362)(5)$, where (174) is the cycle $1 \to 7 \to 4 \to 1$ of length three, and so on. The set of all cycle lengths of π, arranged in nonincreasing order as $(\alpha_1, \alpha_2, \dots)$, form a partition of n called the *cycle type* of π. For example, the cycle type of $[7361524]$ is $(3, 3, 1)$. Two permutations having the same cycle type are conjugate in $\mathfrak{S}_n$ and *vice versa*, so the conjugacy classes of $\mathfrak{S}_n$ are labeled by partitions of n.

The irreducible characters of any finite group are constant on conjugacy classes. In the case of the symmetric group we write $\chi^\lambda(\alpha)$ for the value of the irreducible character of $\mathfrak{S}_n$ labeled by λ evaluated at any permutation with cycle type α. With this notation, the Murnaghan-Nakayama rule ([4, I.7 Ex. 5] or [5, Section 7.17]) reads

$$\chi^\lambda(\alpha) = \sum_{\text{BST } T} (-1)^{\text{ht}(T)}, \tag{2}$$

where the sum is over all border strip tableaux of shape λ and type α. [3]

3. PROOF OF EQUATION (1). We make two simple observations. First, if T is a border strip tableau of shape λ and type α, then reflecting about the diagonal yields a border strip tableau T' of shape λ' and type α. This is clearly bijection between the two sets of border strip tableaux. Second, if B is a border strip of T with hook point u, and B' is the corresponding border strip of T' with hook point u', then $|\ell(u')| = |a(u)|$. Hence

$$\text{ht}(B) + \text{ht}(B') = |\ell(u)| + |a(u)| = |h(u)| - 1 = |B| - 1,$$

and

$$\text{ht}(T) + \text{ht}(T') = \sum_{B \in T} (|B| - 1) = n - m,$$

where m is the number of border strips in T.

Now, suppose that a permutation $\pi \in \mathfrak{S}_n$ has cycle type $\alpha = (\alpha_1, \alpha_2, \dots)$. Each cycle of length α_i can be written as a product of $\alpha_i - 1$ transpositions; for instance, $(1234) = (12)(23)(34)$. Therefore

$$(-1)^\pi = (-1)^{\sum_i (\alpha_i - 1)} = (-1)^{n-c},$$

where c is the total number of cycles of π. If T is a border strip tableau of type α, then $c = m$. Hence

$$(-1)^\pi (-1)^{\text{ht}(T)} = (-1)^{\text{ht}(T')}.$$

[3] Appearances to the contrary notwithstanding, (2) is independent of the ordering of the elements of α.

Summing both sides over all border strip tableaux of shape λ and type α and using the bijection above, we get

$$(-1)^{\pi} \sum_{\text{BST } T} (-1)^{\text{ht}(T)} = \sum_{\text{BST } T'} (-1)^{\text{ht}(T')}$$

from which (1) follows.

ACKNOWLEDGMENT. I am grateful to the referees, whose suggestions and comments resulted in several improvements to this note.

REFERENCES

[1] Chow, T. Y., Paulhus, J. (2021). Algorithmically distinguishing irreducible characters of the symmetric group. *Electron. J. Comb.* 28(2): #P2.5. doi.org/10.37236/9753.
[2] Goldschmidt, D. (1993). *Group Characters, Symmetric Functions, and the Hecke Algebra*. Providence: American Mathematical Society.
[3] Answer by Joppy at https://math.stackexchange.com/questions/2321531.
[4] Macdonald, I. G. (1995). *Symmetric Functions and Hall Polynomials*, 2nd ed. Oxford: Oxford Univ. Press.
[5] Stanley, R. P. (1999). *Enumerative Combinatorics*, Vol. II. Cambridge: Cambridge Univ. Press.

Professor Emeritus, Department of Physics, California State University, San Bernardino CA 92407
prenteln@csusb.edu

