The Spectrum of the Derangement Graph

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The Spectrum of the Derangement Graph

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  Complete Factorial Symmetric Functions
  A Recurrence Relation for the Eigenvalues

Proof of the Conjecture

Future Directions
Cayley Graphs

- $G$ a finite group
- $S \subseteq G$ a symmetric subset of generators:

$$\{ s \in G : s \in S \implies s^{-1} \in S \}$$

- $\Gamma(G, S)$ a Cayley graph:

$$V(\Gamma) = G$$
$$E(\Gamma) = \{ u \sim v \iff vu^{-1} \in S \}$$

- $\Gamma(G, S)$ is normal if $S$ is closed under conjugation.
The Derangement Graph

- $G = S_n$ the symmetric group on $X = \{1, 2, \ldots, n\}$
- $S = D_n$ the set of derangements (fixed point free permutations) on $X$:
  $$\{\sigma \in S_n : \sigma(x) \neq x, \forall x \in X\}$$
- $\Gamma_n := \Gamma(S_n, D_n)$ the derangement graph
Properties of the Derangement Graph

- $\Gamma_n$ is connected for $n > 3$. $S_n$ generated by adjacent transpositions, and $(k, k + 1)$ can be written as $(1, 2, \ldots, n)^2 \cdot (n, n - 1, \ldots, 1)^2(k, k + 1)$

- $\Gamma_n$ is Hamiltonian. (Eggleton and Wallace, 1985)

- $\alpha(\Gamma_n) = (n - 1)!$. (Deza and Frankl, 1977). Bound achieved by coset of stabilizer of a point. These are the only such maximum independent sets (Cameron and Ku, 2003).

- $\omega(\Gamma_n) = n$. Latin squares!

- $\chi(\Gamma_n) = n$. A normal Cayley graph with $\alpha \omega = |V|$ satisfies $\omega = \chi$ (Godsil, unpublished).
Delsarte-Hoffman Bound

Given any regular graph of degree \( k \) with \( N \) vertices we have the Delsarte-Hoffman bound

\[
\alpha \leq \frac{N}{1 - k/\lambda}
\]

where \( \lambda \) is the least eigenvalue of the (adjacency matrix) of the graph.

For the derangement graph \( N = n! \), \( \alpha = (n - 1)! \), and \( k = D_n = |\mathcal{D}_n| \) so

\[
\lambda \geq \frac{-D_n}{n - 1}
\]

Conjecture (C. Ku)

Equality holds.

Equality would imply the Shannon capacity of \( \Gamma_n \) is \( n \).
The Spectrum of a Normal Cayley Graph

Theorem (Diaconis and Shahshahani, 1981; Babai, 1974)

Let \( \Gamma \) be a normal Cayley graph with adjacency matrix \( A \). Then the eigenvalues of \( A \) are given by

\[
\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)
\]

where \( \chi \) ranges over all the irreducible characters of \( G \). Moreover, the multiplicity of \( \eta_\chi \) is \( \chi(1)^2 \).

Proof.

\[
A_{\sigma \tau} = \begin{cases} 
1 & \text{if } \sigma = s \tau \text{ for some } s \in S, \\
0 & \text{otherwise.}
\end{cases}
\]
Proof continued...

Consider

\[ C := \sum_{s \in S} s \in \mathbb{C}[G] \]

As a linear operator on \( \mathbb{C}[G] \)

\[ C \cdot \tau = \sum_{s \in S} s \tau = \sum_{\sigma \in G} \sigma = \sum_{\sigma \in G} A_{\sigma \tau} \sigma \]

So eigenvalues of \( A \) are eigenvalues of \( C \).
Proof completed

By normality, $C$ is in the center of $\mathbb{C}[G]$, so $C$ is a $\mathbb{C}[G]$ module endomorphism. By Schur’s Lemma, $C$ is a constant, say $c$, on any simple $\mathbb{C}[G]$ module $V$. Let $\rho$ be the irreducible representation afforded by $V$ and $\chi$ its character. Then $\rho(C) = cI$ where $c = \frac{\chi(C)}{\chi(1)} = \sum_{s \in S} \frac{\chi(s)}{\chi(1)}$. As the regular representation decomposes into a direct sum of simple modules with multiplicity equal to dimension, the result follows.
Integrality of Derangement Graph Spectrum

Corollary

*The eigenvalues of the derangement graph are integers.*
**Integer Partitions**

Recall that a partition $\lambda$ of $n$, written $\lambda \vdash n$ or $|\lambda| = n$, is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\sum_i \lambda_i = n$. Its length is $l$ and each $\lambda_i$ is a part of the partition.

Partitions are represented by Ferrers diagrams:

$$(4, 3, 2, 2, 1, 1) \quad \leftrightarrow \quad \begin{array}{cccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }
\end{array}$$

and by multiplicity notation:

$$(4, 3, 2, 2, 1, 1) \quad \leftrightarrow \quad 4^1 3^1 2^2 1^2$$
The Irreducible Characters of $S_n$

- To every permutation $\sigma$ we associate a partition $\nu(\sigma)$, namely its cycle type.
- Conjugation preserves cycle type, so the conjugacy classes of $S_n$ are in bijection with partitions of $n$.
- Hence the irreducible characters $\chi_\lambda$ of $S_n$ are also in bijection with partitions of $n$.
- There exist algorithms to compute the irreducible characters of $S_n$, but there is no simple formula, in general.
- There are formulae for specific characters.
The Standard Representation of $S_n$

Let $V = \{e_1, e_2, \ldots, e_n\}$ be the defining representation of $S_n$:

$$\sigma(e_i) = e_{\sigma(i)}$$

$S_n$ leaves fixed the one dimensional subspace $U$ generated by the vector

$$e_1 + e_2 + \cdots + e_n$$

so $U$ affords the trivial representation (which is clearly irreducible). It turns out that the orthogonal complement $W = U^\perp$ also affords an irreducible representation (of dimension $n-1$) called the standard representation of $S_n$. Thus we have the equivariant decomposition

$$V = U \oplus W$$
The Standard Character of $S_n$

By the properties of characters,

$$\chi_V = \chi_U + \chi_W$$

A moment’s thought shows that

$$\chi_V(\sigma) = \# \text{ fixed points of } \sigma$$

Hence

$$\chi_W(\sigma) = \# \text{ fixed points of } \sigma - 1$$

The eigenvalue of $\Gamma_n$ corresponding to this representation is thus

$$\eta_W = \frac{1}{\chi_W(1)} \sum_{s \in D_n} \chi_W(s) = \frac{-D_n}{n - 1}$$

This is the conjectured least eigenvalue!
### The Spectrum of the Derangement Graph

The Standard Character of $S_n$

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<tr>
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<th>$1^6$</th>
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<th>$2^3$</th>
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The Ring of Symmetric Functions

- $S_n$ acts on elements of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting indices.
- The ring of symmetric functions in $n$ variables is
  \[ \Lambda_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n} \]
- $\Lambda_n$ admits a natural grading into homogeneous pieces of degree $k$.
- There are many bases for $\Lambda_n$. We need two: the homogeneous symmetric functions and the Schur functions.
Complete Homogeneous Symmetric Functions

The homogeneous symmetric functions are

\[ h_{\{\lambda_1, \lambda_2, \ldots, \lambda_l\}} = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l} \]

where \( h_k \) is the sum of all monomials of degree \( k \). For example \((n = 3)\)

\[
\begin{align*}
 h_1 &= x_1 + x_2 + x_3 \\
 h_2 &= x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_2 x_3 + x_1 x_3 \\
\end{align*}
\]

so

\[
\begin{align*}
 h_{(2,1)} &= h_2 h_1 \\
 &= x_1^3 + x_2^3 + x_3^3 + 2x_1^2 x_2 + 2x_1^2 x_3 \\
 &\quad + 2x_2^2 x_3 + 2x_1 x_2^2 + 2x_1 x_3^2 + 2x_2 x_3^2 \\
 &\quad + 3x_1 x_2 x_3 \\
\end{align*}
\]
Young Tableaux

The Schur functions $s_\lambda$ can be defined combinatorially. To every partition $\lambda$ associate a semistandard Young tableau $T$ (SSYT $T$: weakly increasing rows, strictly increasing columns) of shape $\lambda$

$\begin{array}{c}
(4, 2, 1) \\
\end{array} \quad \longleftrightarrow \quad \begin{array}{ccc}
1 & 1 & 2 & 2 \\
2 & 3 \\
4
\end{array}$

The type of $T$ is a vector giving the multiplicities of each entry. In the above example, type $T = (2, 3, 1, 1)$. Associated to a tableau is the monomial $x^{\text{type}}$. In the example,

$$x^T := x_1^2 x_2^3 x_3 x_4$$
(Skew) Schur Functions

Generalize. Let $\mu \subseteq \lambda$ (boxwise at upper left corner). Define a skew SSYT of shape $\lambda/\mu$ by removing the boxes in $\mu$ and filling in what remains

$$(4, 2, 1)/(2, 1) \leftrightarrow \begin{array}{c} 1 \\ 3 \\ 3 \end{array}$$

The tableau monomial $x^T$ is defined as before. The skew Schur function of shape $\lambda/\mu$ is

$$s_{\lambda/\mu} = \sum_T x^T$$

where the sum is over all skew SSYT of shape $\lambda/\mu$. If $\mu = \emptyset$ then $s_\lambda$ is the Schur function of shape $\lambda$
Hall Inner Product and Kostka Numbers

Define the canonical (Hall) inner product on symmetric functions

\[(s_\lambda, s_\mu) = \delta_{\lambda,\mu}\]

It turns out that

\[(s_\lambda, h_\mu) = K_{\lambda,\mu}\]

where \(K_{\lambda,\mu}\) is the Kostka number, namely the number of semistandard Young tableau of shape \(\lambda\) and type \(\mu\).
Stanley’s Theorem

Following Stanley, we define

\[ d_\lambda = \sum_{s \in D_n} \chi_\lambda(s) \]

Theorem (Stanley, EC2)

\[ \sum_{\lambda \vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k h_k 1^{n-k} \]

where \( (n)_k = n!/(n-k)! \) and the partition \( k 1^{n-k} \) means \( k \) followed by \( n-k \) ones.

Proof. Follows from Cauchy identity and Munagghan-Nakayama rule:

\[ s_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma)p_\nu(\sigma) \]
The Spectrum of the Derangement Graph

The Eigenvalues of the Derangement Graph

The Eigenvalues of Derangement Graph

Theorem

The eigenvalues of the derangement graph are given by

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (\binom{n}{k} f_{\lambda/k})$$

Proof. Taking inner product with $s_\lambda$ gives

$$d_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (\binom{n}{k} K_{\lambda,k \, n-k})$$

But

$$K_{\lambda,k \, n-k} = f_{\lambda/k}$$

where $f_{\lambda/\mu}$ is the number of SYT of skew shape $\lambda/\mu$ (SYT = strictly increasing in rows and columns)
Proof completed

Example. $n = 7$, $\lambda = (4, 2, 1)$, $k = 2$:

$$(4, 2, 1)/(2) \leftrightarrow \begin{bmatrix} 1 & 1 & * & * \\ * & * \\ * & * \end{bmatrix}$$

Finally, use theorem on eigenvalues of Cayley graph

$$\eta_{\lambda} = \frac{1}{\chi_{\lambda}(1)}d_{\lambda}$$

and fact that

$$\chi_{\lambda}(1) = f^{\lambda}$$
A More Explicit Form

Define the shifted partition

$$\mu_i := \lambda_i + l - i$$

Also, define

$$A(\mu) := \begin{vmatrix}
(\mu_1)_{l+k-1} & \mu_1^{l-2} & \mu_1^{l-3} & \cdots & 1 \\
(\mu_2)_{l+k-1} & \mu_2^{l-2} & \mu_2^{l-3} & \cdots & 1 \\
(\mu_3)_{l+k-1} & \mu_3^{l-2} & \mu_3^{l-3} & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\end{vmatrix}$$

and

$$\omega_k(\mu_1, \mu_2, \ldots, \mu_l) := \frac{A(\mu)}{\prod_{1 \leq i \leq j \leq l}(\mu_i - \mu_j)}$$
Another Expression for the Eigenvalues

**Theorem**

*The eigenvalues of the derangement graph are given by*

\[ \eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} \omega_k(\mu_1, \mu_2, \ldots, \mu_l) \]

**Proof.** The Frobenius formula and the hook formula.
Complete Factorial Symmetric Functions

Chen and Louck (1993) defined the complete factorial symmetric functions by

$$w_k(z_1, z_2, \ldots, z_n) = \sum \prod_{i_1+i_2+\cdots+i_n=k, 1\leq j\leq n} (z_j-i_1-i_2-\cdots-i_{j-1}-j+1)_{i_j}$$

These generalize the ordinary complete symmetric functions:

$$w_k(z_1, z_2, \ldots, z_n) = h_k(z_1, z_2, \ldots, z_n) + \text{lower order terms}$$

(They are special cases of the shifted Schur functions of Okounkov and Olshanski.)
A Result of Chen and Louck

Chen and Louck show that

\textbf{Theorem}

\[ \omega_k(\mu_1, \mu_2, \ldots, \mu_l) = w_k(\mu_1, \mu_2, \ldots, \mu_l) \]
A Lemma of Verde-Star

Idea of Proof. The key result is the following

Lemma (Verde-Star, 1991)

The divided difference of the falling factorial function is

\[
\frac{(x)_{m+1} - (y)_{m+1}}{x - y} = \sum_{0 \leq k \leq m} (x)_k (y - k - 1)_{m-k}
\]

Iterating this lemma yields the result.
Eigenvalues Again

**Theorem**

The eigenvalues of the derangement graph are given by

\[ \eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} w_k(\mu_1, \mu_2, \ldots, \mu_l) \]

where \( \mu \) is the shifted partition associated to \( \lambda \), \( n = |\lambda| \), and \( w_k(\mu_1, \mu_2, \ldots, \mu_l) \) is the complete factorial symmetric function defined above.

Integrality!
Given a partition, say $\lambda = (4, 3, 2, 1, 1)$ we define two subpartitions:

$\lambda_h \rightarrow \begin{array}{c}
\bullet \\
\bullet
\end{array} =: \lambda - h$

$\lambda - 1 \rightarrow \begin{array}{c}
\bullet \\
\bullet
\end{array} =: \lambda - 1$
The Main Theorem

Theorem

The eigenvalues of the derangement graph satisfy the following recurrence:

\[ \eta_\lambda = (-1)^h \left( \eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1} \right) \]

with initial condition \( \eta_\emptyset = 1 \).

For example (denoting the eigenvalue by the partition):

\[ (-1)^8 \left( \begin{array}{c} \hline \hline \hline \end{array} \right) + (-1)^4 \cdot 8 \cdot \begin{array}{c} \hline \hline \hline \end{array} \]
Proof of Main Theorem

Proof. Follows from the recurrence relation for complete factorial symmetric functions (Chen and Louck):

\[
w_k(z_1, z_2, \ldots, z_n) = w_k(z_2 - 1, z_3 - 1, \ldots, z_n - 1) \\
+ z_1 w_{k-1}(z_1 - 1, z_2 - 1, \ldots, z_n - 1)
\]
Proof of Ku’s Conjecture

Theorem
Ku’s conjecture is true. Moreover, for \( n \geq 5 \) the least eigenvalue of the derangement graph \( \Gamma_n \) is uniquely achieved by the standard representation (namely, the shape \( \lambda = (n - 1, 1) \)).

Proof (outline).

▶ The maximum eigenvalue is achieved by the trivial representation:

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ }\\
\text{ } & \text{ } & \text{ } & \text{ }\\
\end{array} \quad \text{ } \text{ } \begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }\\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }\\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ }\\
\end{array} = D_n
\]

General result.
Proof continued...

- The conjecture holds for hooks.

A calculation.

- The conjecture holds for near hooks

Another calculation.
The conjecture holds for all shapes. By above may assume \( \lambda \) is neither a hook \((n = h)\) nor a near hook \((n = h + 1)\). So we may assume \( n \geq h + 2 \) and \( h > l \geq 2 \). Thus...
Proof concluded

\[ |\eta_\lambda| = |\eta_{\lambda-h} + (-1)^\lambda h\eta_{\lambda-1}| \]
\[ \leq |\eta_{\lambda-h}| + h|\eta_{\lambda-1}| \]
\[ \leq D_{n-h} + hD_{n-l} \]
\[ \leq (1 + h)D_{n-l} \]
\[ \leq (n - 1)D_{n-l} \]
\[ \leq (n - 1)D_{n-2} \]
\[ = D_{n-1} + (-1)^n \]
\[ \leq D_{n-1} + 1 \]
\[ < D_{n-1} + D_{n-2} \]
\[ = \frac{D_n}{n - 1} \]
\[ = |\eta(n-1,1)| \]
Interesting Sequences

- Interesting sequences arise from special cases. Example: staircase shapes

\[
a_m = - [a_{m-2} + (-1)^m (2m - 1) a_{m-1}] \\
0, -1, -5, 36, 329, -3655, \ldots
\]

Sloane’s online encyclopaedia of integer sequences gives this sequence *modulo signs* as \(y(-1)\) where \(y(x)\) is the so-called Bessel polynomial.
A Question

Question: Do the central characters of the symmetric group themselves obey a recurrence?