On the Spectra of Simplicial Rook Graphs

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On the spectra of simplicial rook graphs

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Abstract Martin and Wagner determined the integral eigenvalue spectrum of the simplicial rook graphs on the triangular lattice by explicitly constructing their eigenvectors. In this work we deduce the same result by instead constructing the characteristic polynomials for this class of graphs. The resulting analysis provides a neat explanation for the observed spectral structure.

Keywords simplicial rook graph · spectral graph theory · block matrices

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1 Introduction

Let $C_{n,d}$ denote the set of weak d-compositions of n. That is,

$$C_{n,d} = \{(x_1, \dots, x_d) : \sum_{i=1}^{d} x_i = n \text{ and } x_i \geq 0 \text{ for } 1 \leq i \leq d\}.$$

We define a graph $S(n, d)$ on $C_{n,d}$ by joining two compositions if they differ in precisely two entries. The graph $S(n, d)$ is called a *simplicial rook graph*, because the vertices can be identified with the lattice points inside the n^{th} dilate of the standard simplex, where two points are joined by an edge if they lie upon the same lattice line. The terminology arises because we may view two points joined by an edge as a pair of rooks on a simplicial chessboard. In particular, the number of non-attacking rooks on such a chessboard is the independence number of the graph.

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Simplicial rook graphs were first defined and studied in a very nice paper by Martin and Wagner [5], although the independence number of $S(n,3)$, namely $\lfloor(2n+3)/3\rfloor$, was obtained earlier (and independently) by Blackburn, Paterson, and Stinson [2] and Nivasch and Lev [6]. (See also the discussion in [4].) It is not difficult to see ([5], Prop. 2.1) that $S(n,d)$ has $\binom{n+d-1}{n}$ vertices and is regular of degree $n(d-1)$. Martin and Wagner determined the spectrum of $S(n,3)$ for all n, and, on the basis of computational evidence, conjectured that the spectra of all the graphs $S(n,d)$ are integral. They also conjectured that the least eigenvalue of $S(n,d)$ is equal to $\max\{-n, -\binom{d}{2}\}$, and gave conjectured values for some of their multiplicities. [1] All of these conjectures were subsequently confirmed by some very clever arguments in a paper of Brouwer, Cioabă, Haemers, and Vermette [3], who also proved several other interesting results about simplicial rook graphs. (See also [7].) For more recent results on the independence numbers of simplicial rook graphs, see [1].

Many questions about these graphs remain unanswered, such as the general eigenvalue spectrum of $S(n,d)$ and the exact independence number of $S(n,d)$. These appear to be difficult problems, especially because, as was pointed out by Martin and Wagner, these graphs do not seem to have nice graph theoretical characterizations; in particular, they are generally neither vertex-transitive nor distance-regular.

We do not solve these problems here. Instead, we revisit the computation of the eigenvalue spectra of the simplicial rook graphs with $d=3$. As noted above, this problem was already solved by Martin and Wagner, who observed that the eigenvectors of $S(n,3)$ can be given a nice geometric interpretation in terms of lines and hexagons. One feature of the spectra of the graphs $S(n,3)$ is their dependence on the parity of n, which appears somewhat mysterious in the treatment of [5]. The purpose of this work is to derive the characteristic polynomials of the graphs $S(n,3)$, thereby providing a simpler explanation for the observed structure of the eigenvalue spectra, including their modular dependence. Along the way we discover a matrix whose eigenanalysis may be of independent interest. We end with some remarks on the higher dimensional cases.

2 Simplicial lattice lines

To find the spectra of the simplicial rook graphs $S(n,3)$, we will exploit the fact that the lattice lines have a particularly simple intersection pattern when $d=3$. Let $A(n,d)$ be the adjacency matrix of $S(n,d)$. Fix a pair (i,j) with $1 \le i < j \le d$ and an integer k with $0 \le k \le n$. Fix a weak composition

[1] Least eigenvalues are of interest in their own right, but also because they are connected to the independence number α of a graph *via* the Hoffman bound, which states that $\alpha \le |V|/(1-k/\tau)$, where $|V|$ is the number of vertices, k is the degree, and τ is the least eigenvalue. But the Hoffman bound is not always exact. For instance, for $S(n,3)$ with $n>3$, the Hoffman bound gives $\alpha < 3(n+2)(n+1)/(4n+6)$, which is weaker than the actual bound given above.

$\alpha = (\alpha_1, \ldots, \underbrace{0}_{i}, \ldots, \underbrace{0}_{j}, \ldots, \alpha_d)$ with *weight* $|\alpha| := \sum_s \alpha_s = n - k$. Define the *line at α in the direction* (i, j) by [2]

$$L_{ij}^{\alpha} = \{(\alpha_1, \ldots, \underbrace{m}_{i}, \ldots, \underbrace{k-m}_{j}, \ldots, \alpha_d) : 0 \le m \le k\} \subset C_{n,d}.$$

We also denote this line by

$$L_{ij}^{(\alpha_1,\ldots,\alpha_{i-1},*,\alpha_{i+1},\ldots,\alpha_{j-1},*,\alpha_{j+1},\ldots,\alpha_d)}$$

where the stars represent the pairs $(0,k), (1,k-1), \ldots, (k,0)$. Write $|L|$ for the cardinality of L. Observe that the line L_{ij}^{α} could contain only a single point of $S(n,d)$. For instance, when $d = 3$, the line $L_{12}^{(*,*,n)}$ contains only $(0,0,n)$ itself. We require these degenerate lines in what follows.

From the above construction we deduce that the total number of lines of the form L_{ij}^{α} is

$$\begin{aligned} \ell(n,d) :&= \binom{d}{2} \sum_{k=0}^{n} \binom{n-k+d-3}{n-k} \\ &= \binom{d}{2} \sum_{k=0}^{n} \binom{k+d-3}{k} \\ &= \binom{d}{2} \binom{n+d-2}{n}, \end{aligned} \tag{1}$$

as there are $\binom{d}{2}$ choices for the pair $i < j$, and $\binom{n-k+d-3}{n-k}$ weak compositions of $n-k$ into $d-2$ parts.

If we were to write $p(n,d)$ for the number of points of $S(n,d)$, then the previous equation could also be written

$$\ell(n,d) = \binom{d}{2} p(n, d-1). \tag{2}$$

This can also be explained geometrically. Let $\Delta \in \mathbb{R}^d$ be the regular simplex whose vertices are the permutations of $(n, 0, \ldots, 0)$. Assign labels to the facets according to the prescription

$$F_i := \{(x_1, \ldots, x_d) \in \Delta : x_i = 0\}.$$

Every line of the form L_{ij}^{α} is uniquely specified by its direction and by its intersection point with the facet F_i. As there are $\binom{d}{2}$ directions, and $p(n, d-1)$ points on any facet, (2) follows.

[2] Remarks. (1) We refer to L_{ij}^{α} as a 'line', although technically it is just a finite set of points lying along a lattice line. (2) The lines L_{ij}^{α} are cliques of $S(n,d)$, but we prefer the more geometric language in this context.

Write $x \in L$ if the point x is contained in the line L. Let Q be the incidence matrix of points versus lines, so that

$$Q_{xL} = \begin{cases} 1, & \text{if } x \in L, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

By definition, two points are adjacent in $S(n,d)$ if there exists a line passing through both of them. Moreover, because we permit degenerate lines, each point is contained in exactly $\binom{d}{2}$ lines. Hence,

$$A = QQ^T - \binom{d}{2} I. \tag{3}$$

We note in passing that, as QQ^T is positive definite, (3) implies that the minimum eigenvalue of $A(n,d)$ is at least $-\binom{d}{2}$.

We wish to find the spectrum of A, or, equivalently, that of $Y := QQ^T$, whose eigenvalues differ from those of A by the constant $\binom{d}{2}$. We use a standard fact from linear algebra that the spectrum of QQ^T is identical to the spectrum of $M := Q^T Q$ up to zeros. By construction,

$$M_{L,L'} = |L \cap L'|. \tag{4}$$

In the next section we construct M explicitly when $d = 3$, and in the following section we find its spectrum.

3 The case $d = 3$

Now fix $d = 3$. Order the lines in triplets, as follows:

$$\left.\begin{array}{rl} 1) & L_{12}^{(*,*,n)}, L_{13}^{(*,n,*)}, L_{23}^{(n,*,*)}, \\ 2) & L_{12}^{(*,*,n-1)}, L_{13}^{(*,n-1,*)}, L_{23}^{(n-1,*,*)}, \\ & \vdots \\ n) & L_{12}^{(*,*,1)}, L_{13}^{(*,1,*)}, L_{23}^{(1,*,*)}, \\ n+1) & L_{12}^{(*,*,0)}, L_{13}^{(*,0,*)}, L_{23}^{(0,*,*)}. \end{array}\right\} \tag{5}$$

A line L contained in a triplet i contains i points and has weight $w(L) = n-i+1$ (corresponding to the fixed entry of the superscript). Evidently, M will be a $3(n+1) \times 3(n+1)$ matrix.

Define two lines L and L' to be *parallel* if they share the same subscript (for then the corresponding lattice lines are parallel in the geometric sense.) If $L \neq L'$, they do not meet. If $L = L'$, then they meet in $|L|$ points.

If the lines are not parallel (written $L \nparallel L'$), then either they do not meet, or they intersect in one point. Consider, for instance, $L_{12}^{(*,*,\ell)}$ and $L_{13}^{(*,k,*)}$. The only way for them to meet is to share a point of the form (j,k,ℓ) with

$j = n - k - \ell$. Such a point will exist on each line whenever $\ell + k \leq n$, which is to say, whenever $w(L_{12}^{(*,*,\ell)}) + w(L_{13}^{(*,k,*)}) \leq n$. Similar considerations apply to intersections of the form $L_{12} \cap L_{23}$ and $L_{13} \cap L_{23}$. Therefore, in general we have

$$M_{LL'} = |L \cap L'| = \begin{cases} |L|, & \text{if } L = L', \\ 1, & \text{if } L \nparallel L' \text{ and } w(L) + w(L') \leq n, \text{ and} \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

It follows that the entries of the matrix M depend only on the triplet to which each line belongs, and whether or not the lines are parallel. Thus, the matrix M naturally breaks up into 3×3 sized blocks M_{ij}, where i and j label the triplets. Let L belong to triplet i and L' belong to triplet j, so that $w(L) = n - i + 1$ and $w(L') = n - j + 1$.

Consider first the case $i = j$. If $L = L'$, then $(M_{ii})_{LL} = i$. If $L \nparallel L'$, there are two cases, depending on the weights of the lines. If $2(n - i + 1) \leq n$, or equivalently, $2i - 2 \geq n$, then $(M_{ii})_{LL'} = 1$, else $(M_{ii})_{LL'} = 0$. That is,

$$(M_{ii})_{LL'} = \begin{cases} i, & \text{if } L = L', \\ 1, & \text{if } L \nparallel L' \text{ and } 2i - 2 \geq n, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let I denote the 3×3 identity matrix, J the 3×3 all-ones matrix, and set

$$K := J - I$$

(the adjacency matrix of the 3×3 complete graph). Define $\Xi(P) = 1$ if the proposition P is true, and $\Xi(P) = 0$ if P is false. Then, with the ordering of lines given in (5), we may write

$$M_{ii} = iI + K\Xi(2i - 2 \geq n), \tag{7}$$

If $i \neq j$, then the same ordering of lines yields

$$M_{ij} = K\Xi(i + j - 2 \geq n). \tag{8}$$

(It is these dependencies on n that ultimately explains why the spectrum of $S(n, 3)$ depends upon the parity of n.) Putting everything together, we can write M elegantly in block matrix form as

$$M = \begin{pmatrix} I & 0 & 0 & \cdots & 0 \\ 0 & 2I & 0 & \cdots & 0 \\ 0 & 0 & 3I & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (n+1)I \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & K \\ 0 & 0 & \cdots & K & K \\ 0 & \cdots & K & K & K \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ K & K & K & \cdots & K \end{pmatrix}. \tag{9}$$

In the next section we exploit the simple form of this matrix to find its spectrum.

4 The eigenvalues of M

Define the following two $(n+1) \times (n+1)$ matrices:

$$X := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n+1 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & \cdots & 1 & 1 & 1 \\ \vdots & \iddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

When $n = 0$ we set $T = (1)$. Then we can write (9) compactly as

$$M = X \otimes I + T \otimes K, \tag{10}$$

where $A \otimes B$ is the Kronecker product of A and B. We wish to find the characteristic polynomial $\chi_M(x) := \det(M - xI)$ of M. To do so, we proceed as follows.

The matrix K is clearly diagonalizable, with characteristic polynomial

$$\chi_K(\mu) = -(\mu - 2)(\mu + 1)^2. \tag{11}$$

Let v be an eigenvector of K with eigenvalue μ, so that $Kv = \mu v$. Next, observe that the matrix

$$B(\mu) := X + \mu T$$

is real and symmetric, so also diagonalizable. Let u be an eigenvector of $B(\mu)$ with eigenvalue λ, so that $B(\mu)u = \lambda u$. Then from (10) we have

$$M(u \otimes v) = Xu \otimes v + Tu \otimes \mu v = B(\mu)u \otimes v = \lambda(u \otimes v).$$

Thus, the eigenvalues of M may be found by computing the eigenvalues of $B(\mu)$ for each value of μ. [3] As the roots of (11) are $\mu = \{-1, -1, 2\}$, the spectrum of M consists of the eigenvalues of $B(-1)$ (each with multiplicity two), together with the eigenvalues of $B(2)$ (each with multiplicity one). We summarize these results in the following.

Proposition 1 *For any matrix C, let $\Sigma(C)$ denote the multiset of eigenvalues of C. Then, with the definitions above,*

$$\Sigma(M) = \biguplus_{\mu \in \{-1,-1,2\}} \Sigma(B(\mu)),$$

where $\uplus$ denotes multiset union. Equivalently, the characteristic polynomial of M is given (possibly up to an overall constant) by

$$\chi_M(x) \sim (\chi_{B(-1)}(x))^2 \chi_{B(2)}(x).$$

(Here, '$\sim$' denotes proportionality.)

[3] We may also obtain the eigenvectors of M by letting u and v vary over the eigenvectors of K and $B(\mu)$, respectively. It would be interesting to relate these to the hexagon and line eigenvectors used in [5].

The characteristic polynomial of $B(\mu)$ is

$$\chi_{B(\mu)}(x) := \det(-xI + X + \mu T).$$

As T is invertible, and we may factor it out. Set

$$r := n + 1, \tag{12}$$

and define

$$H_\mu(r; x) := (-xI + X)T^{-1} + \mu I, \tag{13}$$

so that $H_\mu(r, x)T = -xI + B_\mu$. Inverting T and multiplying, we obtain $H_\mu(1, x) = -x + 1 + \mu$, and, for $r > 1$,

$$H_\mu(r; x) = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 & x-1 & -(x-1) \\ 0 & \mu & 0 & \cdots & x-2 & -(x-2) & 0 \\ \vdots & \vdots & \ddots & \ddots & \iddots & \vdots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \ddots & \vdots & \vdots \\ x-r+1 & -(x-r+1) & 0 & \cdots & \cdots & \mu & 0 \\ -(x-r) & 0 & \cdots & \cdots & \cdots & \cdots & \mu \end{pmatrix}. \tag{14}$$

Set

$$p_\mu(r; x) := \det H_\mu(r; x), \tag{15}$$

so that

$$\chi_{B(\mu)}(x) = p_\mu(r; x) \det T. \tag{16}$$

It is easy to see (by row operations bringing T to upper triangular form) that $\det T = \pm 1$. As we are only interested in the zeros of $\chi_{B(\mu)}(x)$, it suffices to compute $p_\mu(r; x)$ up to an overall constant.

There are two approaches. Using row and column operations, one can show that $p_\mu(r; x)$ factors nicely. We sketch this technique in two special cases, which suffice to illustrate the general method. The second approach is to derive a recurrence relation for $p_\mu(r; x)$. Although the factorization property of $p_\mu(r, x)$ is not obvious from the recurrence itself, examination of the data will permit us to deduce its factored form.

4.1 Row and column operations

We first consider the case $\mu = -1$, and illustrate the general case using the example of $r = 6$. From (14) and (15) we have

$$p_{-1}(6; x) \sim \begin{vmatrix} 1 & 0 & 0 & 0 & -(x-1) & x-1 \\ 0 & 1 & 0 & -(x-2) & x-2 & 0 \\ 0 & 0 & -(x-4) & x-3 & 0 & 0 \\ 0 & -(x-4) & x-4 & 1 & 0 & 0 \\ -(x-5) & x-5 & 0 & 0 & 1 & 0 \\ x-6 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

In what follows we write 'R' and 'C' to denote the rows and columns of the matrix, and we ignore overall signs. In the first step, we factor out an $x-4$ term from C3 to get

$$p_{-1}(6;x) \sim (x-4)\begin{vmatrix} 1 & 0 & 0 & 0 & -(x-1) & x-1 \\ 0 & 1 & 0 & -(x-2) & x-2 & 0 \\ 0 & 0 & 1 & x-3 & 0 & 0 \\ 0 & -(x-4) & -1 & 1 & 0 & 0 \\ -(x-5) & x-5 & 0 & 0 & 1 & 0 \\ x-6 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Do R3+R4→R4 to get

$$p_{-1}(6;x) \sim (x-4)\begin{vmatrix} 1 & 0 & 0 & 0 & -(x-1) & x-1 \\ 0 & 1 & 0 & -(x-2) & x-2 & 0 \\ 0 & 0 & 1 & x-3 & 0 & 0 \\ 0 & -(x-4) & 0 & x-2 & 0 & 0 \\ -(x-5) & x-5 & 0 & 0 & 1 & 0 \\ x-6 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Use C3 to clear R3:

$$p_{-1}(6;x) \sim (x-4)\begin{vmatrix} 1 & 0 & 0 & 0 & -(x-1) & x-1 \\ 0 & 1 & 0 & -(x-2) & x-2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -(x-4) & 0 & x-2 & 0 & 0 \\ -(x-5) & x-5 & 0 & 0 & 1 & 0 \\ x-6 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Notice that this eliminates $x-3$ from the matrix. Rather than write out all the subsequent matrices, we abbreviate the next steps as follows:

- Factor out $x-2$ from C4.
- Do R4+R2→R2.
- Clear R4.
- Factor out $x-5$ from C2.
- Do R2+R5→R5.
- Clear R2.
- Factor out $x-1$ from C5.
- Do R5+R1→R1.
- Clear R5.
- Factor out $x-6$ from C1.
- Do R1+R6→R6.
- Factor out x from R6.

The result is

$$p_{-1}(6;x) \sim x(x-1)(x-2)(x-4)(x-5)(x-6).$$

Clearly, this method works in general. Moreover, it is evident that $p(x)$ factors, and it is clear why the result depends on the parity of n: if r is even

(n is odd), then the final result will be missing a factor of $x - \frac{r}{2}$, but if r is odd (n is even) then the final result will be missing a factor of $x - \frac{r+1}{2}$. The general result (which we also prove below) is

$$p_{-1}(r;x) \sim \begin{cases} \dfrac{1}{x - r/2} \displaystyle\prod_{i=0}^{r}(x-i), & \text{if } r \text{ is even, and} \\ \dfrac{1}{x-(r+1)/2} \displaystyle\prod_{i=0}^{r}(x-i), & \text{if } r \text{ is odd.} \end{cases} \tag{17}$$

The case of $\mu = 2$ is a little more tricky. Now we have

$$p_2(6;x) = \begin{vmatrix} 2 & 0 & 0 & 0 & x-1 & -(x-1) \\ 0 & 2 & 0 & x-2 & -(x-2) & 0 \\ 0 & 0 & x-1 & -(x-3) & 0 & 0 \\ 0 & x-4 & -(x-4) & 2 & 0 & 0 \\ x-5 & -(x-5) & 0 & 0 & 2 & 0 \\ -(x-6) & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

This time, multiply R3 by $(x-4)$ and R4 by $(x-1)$ then do R3+R4→R4 to clear the (4,3) entry. To avoid changing the determinant, we have to divide the determinant by $(x-1)(x-4)$. [4] This gives

$$p_2(6;x) \sim ((x-1)(x-4))^{-1} \times \begin{vmatrix} 2 & 0 & 0 & 0 & x-1 & -(x-1) \\ 0 & 2 & 0 & x-2 & -(x-2) & 0 \\ 0 & 0 & (x-1)(x-4) & -(x-3)(x-4) & 0 & 0 \\ 0 & (x-4)(x-1) & 0 & -(x-2)(x-7) & 0 & 0 \\ x-5 & -(x-5) & 0 & 0 & 2 & 0 \\ -(x-6) & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

But now multiply C3 by $((x-1)(x-4))^{-1}$ to normalize the (3,3) entry. This allows us to clear R3:

$$p_2(6;x) \sim \begin{vmatrix} 2 & 0 & 0 & 0 & x-1 & -(x-1) \\ 0 & 2 & 0 & x-2 & -(x-2) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & (x-4)(x-1) & 0 & -(x-2)(x-7) & 0 & 0 \\ x-5 & -(x-5) & 0 & 0 & 2 & 0 \\ -(x-6) & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

[4] We need not worry about zero divisors as long as we treat 'x' as an indeterminate. In any case, the denominators will all clear in the end.

Now repeat this trick by removing a factor of $(x-7)^{-1}$ from the determinant, then multiply R2 by $(x-7)$, and do R2+R5→R2 to get

$$p_2(6;x) \sim (x-7)^{-1} \begin{vmatrix} 2 & 0 & 0 & 0 & x-1 & -(x-1) \\ 0 & (x-5)(x+2) & 0 & 0 & -(x-2)(x-7) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & (x-4)(x-1) & 0 & -(x-2)(x-7) & 0 & 0 \\ x-5 & -(x-5) & 0 & 0 & 2 & 0 \\ -(x-6) & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

Now remove a factor of $(x-2)(x-7)$ (up to sign) and clear R4 to get

$$p_2(6;x) \sim (x-2) \begin{vmatrix} 2 & 0 & 0 & 0 & x-1 & -(x-1) \\ 0 & (x-5)(x+2) & 0 & 0 & -(x-2)(x-7) & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ x-5 & -(x-5) & 0 & 0 & 2 & 0 \\ -(x-6) & 0 & 0 & 0 & 0 & 2 \end{vmatrix}$$

Continuing in this fashion, we eventually obtain

$$p_2(6;x) \sim x(x-1)(x-2)(x-5)(x-6)(x-13).$$

Evidently, $p_2(6;x)$ factors. Moreover, we again expect the final result to depend on the parity of r. But deducing the general form of $p_2(r;x)$ from this procedure is more difficult than in the $\mu = -1$ case. Instead, we will do so in the following section.

4.2 The recurrence

Theorem 1 *The following recurrence holds:*

$$p_\mu(r;x) = \mu p_\mu(r-1;r-x) - (x-1)(x-r)p_\mu(r-2;x-1) \qquad (r \geq 2), \tag{18}$$

with initial conditions $p_\mu(0;x) = 1$ *and* $p_\mu(1;x) = -x+1+\mu$.

Proof The initial condition $p_\mu(0;x) = 1$ is set by *fiat* to make the recurrence work, while the second initial condition follows directly from (13). As for the recurrence itself, we expand by the last column of $H_\mu(r;x)$ to get

$$p_\mu(r;x) = \mu \det H_\mu(r;x)[r|r] + (-1)^r(x-1)\det H_\mu(r;x)[1|r]. \tag{19}$$

where $A[i|j]$ denotes the matrix obtained from A by removing the i^{th} row and j^{th} column. To identify the first term in (19), write $C := H_\mu(n;x)[r|r]$. Let P be the permutation matrix representing the permutation $[r-1, r-2, \ldots, 1]$ (in the defining representation). Then PCP^{-1} (or equivalently, PCP, as $P^2 = 1$) is the matrix obtained from C by flipping it horizontally and vertically. Inspection reveals that $PCP = H_\mu(r-1;r-x)$. But $\det PCP = \det C$. For the second term in (19), just expand by the last row of $H_\mu(r;x)[1|r]$. This gives $\det H_\mu(r;x)[1|r] = (-1)^{r-1}(x-r)\det H_\mu(r-2;x-1)$.

Running out the recurrence, we obtain the following data for $1 \leq r \leq 5$:

r	$p_{-1}(r;x)$	$p_2(r;x)$
1	$-x$	$-x+3$
2	$-x(x-2)$	$-x(x-5)$
3	$x(x-1)(x-3)$	$x(x-3)(x-7)$
4	$x(x-1)(x-3)(x-4)$	$x(x-1)(x-4)(x-9)$
5	$-x(x-1)(x-2)(x-4)(x-5)$	$-x(x-1)(x-4)(x-5)(x-11)$
6	$-x(x-1)(x-2)(x-4)(x-5)(x-6)$	$-x(x-1)(x-2)(x-5)(x-6)(x-13)$

These are enough data to make a guess as to the form of $p_\mu(r;x)$. Of course, from (17) we already know what to expect when $\mu = -1$ (at least, up to sign):

Proposition 2 *We have $p_{-1}(0;x) = 1$ by definition, and for $r > 0$,*

$$p_{-1}(r;x) = \begin{cases} (-1)^m \dfrac{1}{x-m} \displaystyle\prod_{i=0}^{2m} (x-i), & \textit{if } r = 2m, \textit{ and} \\ (-1)^{m+1} \dfrac{1}{x-(m+1)} \displaystyle\prod_{i=0}^{2m+1} (x-i), & \textit{if } r = 2m+1. \end{cases}$$

Proof By induction on r. We have $p_{-1}(0;x) = 1$ by definition, and $p_{-1}(1;x) = -x$ from the formula above with $r = 1$. Then, plugging into the recurrence (18) gives, for $r = 2m \geq 2$,

$$\begin{aligned} p_{-1}(2m;x) &= -p_{-1}(2m-1;2m-x) - (x-1)(x-2m)p_{-1}(2m-2;x-1) \\ &= \frac{(-1)^m}{m-x} \prod_{i=0}^{2m-1} (2m-x-i) \\ &\qquad - (x-1)(x-2m)(-1)^{m-1} \frac{1}{x-m} \prod_{i=0}^{2m-2} (x-i-1) \\ &= (1-(x-1)) \frac{(-1)^{m+1}}{x-m} \prod_{i=1}^{2m} (x-i) \\ &= \frac{(-1)^m}{x-m} \prod_{i=1}^{2m} (x-i). \end{aligned}$$

A similar calculation holds for $r = 2m+1$.

Proposition 3 *We have*

$$p_2(r;x) = \begin{cases} (-1)^m \dfrac{x-(4m+1)}{(x-(m+1))(x-m)} \displaystyle\prod_{i=0}^{2m} (x-i), & \textit{if } r = 2m, \textit{ and} \\ (-1)^{m+1} \dfrac{x-(4m+3)}{(x-(m+1))(x-m)} \displaystyle\prod_{i=0}^{2m+1} (x-i), & \textit{if } r = 2m+1. \end{cases}$$

Proof This is similar to the previous proof, but more involved, and is omitted.

Combining Propositions 1, 2, and 3, and using (12) and (16), we arrive at the following.

Theorem 2 *Up to sign, the characteristic polynomial of M is given (for $m \geq 1$) by*

$$\chi_M(x) \sim \begin{cases} \dfrac{x-(4m+3)}{(x-(m+1))^3(x-m)} \displaystyle\prod_{i=0}^{2m+1}(x-i)^3, & \text{if } n = 2m, \text{ and} \\ \dfrac{x-(4m+1)}{(x-(m+1))(x-m)^3} \displaystyle\prod_{i=0}^{2m}(x-i)^3, & \text{if } n = 2m-1. \end{cases}$$

5 The spectrum of $S(n, 3)$

Recall that the incidence matrix Q of points versus lines is $p \times \ell$. When $d = 3$ we have $p = \binom{n+2}{2} = \frac{1}{2}(n+2)(n+1)$ and (from Equation (1) or Section 3), $\ell = 3(n+1)$. With $Y = QQ^T$ and $M = Q^T Q$ we have

$$x^\ell \chi_Y(x) = x^p \chi_M(x).$$

From Theorem 2 we see that zero appears exactly thrice in the spectrum of M, for any $n \geq 1$. Hence, the multiplicity of zero as a root of $\chi_Y(x)$ is

$$p + 3 - \ell = \frac{1}{2}(n+2)(n+1) + 3 - 3(n+1) = \binom{n-1}{2}.$$

By virtue of (3), the eigenvalues of $S(n, 3)$ are just those of Y, shifted down everywhere by $\binom{3}{2} = 3$. This brings us to the final result.

Theorem 3 *Up to sign, the characteristic polynomial of $S(n, 3)$ is given (for $m \geq 1$) by*

$$\begin{cases} (x+3)^{\binom{2m-1}{2}} \dfrac{x-4m}{(x-(m-2))^3(x-(m-3))} \displaystyle\prod_{i=1}^{2m+1}(x-i+3)^3, & \text{if } n = 2m, \text{ and} \\ (x+3)^{\binom{2m-2}{2}} \dfrac{x-(4m-2)}{(x-(m-2))(x-(m-3))^3} \displaystyle\prod_{i=1}^{2m}(x-i+3)^3, & \text{if } n = 2m-1. \end{cases}$$

This reproduces the main result (Theorem 1.1) of [5], and explains the regularity of the observed spectral structure.

6 Higher dimensions

The methods employed here for $d = 3$ do not extend easily to higher values of d. The authors of [5] and [3] did obtain some interesting partial results for the spectra of $S(n, d)$ for arbitrary n and d, but the general problem remains stubbornly out of reach. Even the case $d = 4$ is not known definitively, although Brouwer *et al* put forward a conjecture ([3], Section 12) for the answer in this case. Just as the spectra of the graphs $S(n, 3)$ depend on the parity of n, so do the spectra of the graphs $S(n, 4)$ depend on congruence classes of n modulo 2, 4, and 6. The appearance of more complicated modular dependencies suggests that the general answer will not be simple.

References

1. A. Ahadi, M. Mollahajiaghaei, and A. Dehghan, "On the maximum number of non-attacking rooks on a high-dimensional simplicial chessboard", *Graphs and Combinatorics* **38** (2022) 52.
2. S.R. Blackburn, M.B. Paterson, and D.R. Stinson, "Putting dots in triangles", *J. Combin. Math. Combin. Comput.* **78** (2011) 23-32.
3. A.E. Brouwer, S.M. Cioabă, W.H. Haemers, and J.R. Vermette, "Notes on simplicial rook graphs", *J. Algebr. Comb.* **43** (2016) 783-799.
4. https://mathoverflow.net/questions/103540/hexagonal-rooks
5. J.L. Martin and J.D. Wagner, "On the spectra of simplicial rook graphs", *Graphs and Combinatorics* **31** (2015) 1589-1611.
6. G. Nivasch and E. Lev, "Nonattacking queens on a triangle", *Math. Mag.* **78(5)** (2005) 399-403.
7. J.R. Vermette, *Spectral and combinatorial properties of friendship graphs, simplicial rook graphs, and extremal expanders.* Ph.D. thesis, University of Delaware, 2015.

Declarations

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