Lectures on Fourier and Laplace Transforms

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1 Fourier Series

1.1 Historical Background

Waves are ubiquitous in nature. Consequently, their mathematical description has been the subject of much research over the last 300 years. Famous mathematicians such as Daniel Bernoulli, Jean D'Alembert, and Leonhard Euler all worked on finding solutions to the wave equation.

One of the most basic waves is a **harmonic** or **sinusoidal** wave. Very early on, Bernoulli asserted that any wave on a string tied between two points $-\pi$ and π (as in Figure 1) could be viewed as an infinite sum of harmonic waves:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx. \qquad (1.1)$$

In other words, harmonic waves are the building blocks of all waves.

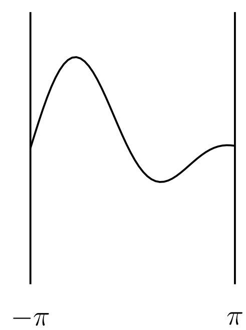

Figure 1: A string from $-\pi$ to π

As is usually the case with these things, the problem turned out to be a bit more subtle than was first thought. In particular, in the beginning it suffered from a lack of rigour. In 1807, Jean Baptiste Joseph Fourier submitted a paper to L'Academie des Sciences de Paris in which he asserted that essentially every function on $[-\pi, \pi]$ (and not just those with fixed ends) is representable as an infinite sum of harmonic functions. Today we call this the **Fourier series expansion of the function** $f(x)$**:**

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)} \tag{1.2}$$

where the coefficients a_n and b_n may be obtained from the function $f(x)$ in a definite way. Although Fourier was not the first person to obtain formulae for a_n and b_n (Clairaut and Euler had already obtained these formulae in 1757 and 1777, respectively), Fourier was able to show that the series worked for almost all functions, and that the coefficients were equal for functions that agree on the interval.

Fourier's paper was rejected by a powerhouse trio of mathematicians, Lagrange, Laplace, and Legendre, as not being mathematically rigorous enough. Fortunately, Fourier was not dissuaded, and finally in 1822 he published one of the most famous scientific works in history, his *Théorie Analytique de la Chaleur*, or *Analytic Theory of Heat*. Fourier's book had an immediate and profound impact on all of science, from Mathematics to Physics to Engineering, and led to significant advances in our understanding of nature. The essence of the idea is this:

> Fourier series are a tool used to represent arbitrary functions as a sum of simple ones.

Two questions immediately present themselves. First, given a function $f(x)$, how do we obtain the **Fourier coefficients** a_n and b_n? This is the **problem of inversion**. Second, suppose you have the Fourier coefficients, and you assemble them into the infinite sum appearing on the right in (1.2). How do you know that this sum will give you back the function you started with? This is the **problem of convergence**. It turns out that the first problem is easily solved, whereas the solution to the second problem is quite involved. For this reason, we will consider these questions in order. But first we need to generalize our understanding of vectors.

1.2 Inner Product Spaces

By now you are quite familiar with vectors in $\mathbb{R}^n$. A vector $\boldsymbol{u}$ in $\mathbb{R}^n$ is just a collection of numbers $u = (u_1, u_2, \ldots, u_n)$. We define the sum of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ by

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n), \tag{1.3}$$

and the product of a vector $\boldsymbol{u}$ and a real number a by

$$a\boldsymbol{u} = (au_1, au_2, \ldots, au_n). \tag{1.4}$$

These operations allow us to define the **linear combination** $a\boldsymbol{u} + b\boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. We say that $\mathbb{R}^n$ forms a vector space over $\mathbb{R}$, the real numbers.

In what follows, it is convenient to consider other kinds of vector spaces, so a general definition would come in handy. Here it is. [1]

Definition. *A* **vector space** V *over a field* $\mathbb{F}$ *is a set of objects* $u, v, w, \cdots \in V$*, called vectors, that is closed under linear combinations, with scalars taken*

[1] Actually, there are a few other properties, but basically they all follow from the one property given in the definition. Incidentally, a **field** is basically just a set of numbers that you can add, subtract, multiply, and divide. We will only be concerned with two fields, namely $\mathbb{R}$, the real numbers, and $\mathbb{C}$, the complex numbers.

from $\mathbb{F}$. *That is,*

$$u, v \in V \text{ and } a, b \in \mathbb{F} \Rightarrow au + bv \in V. \tag{1.5}$$

As we have seen, $\mathbb{R}^n$ is a vector space over $\mathbb{R}$. Now consider the set $\mathcal{F}$ of all real functions on the interval $[-\infty, \infty]$. We define the sum of two functions f and g by

$$(f + g)(x) = f(x) + g(x) \tag{1.6}$$

(this is called *pointwise addition*), and the product of a scalar and a function by

$$(af)(x) = af(x). \tag{1.7}$$

This makes $\mathcal{F}$ into a vector space over $\mathbb{R}$.

Recall that a **basis** of a vector space V is a collection of vectors that is **linearly independent** and that **spans** the space. The **dimension** of the space is the number of basis elements. The standard basis of $\mathbb{R}^n$ consists of the vectors $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{e}_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\boldsymbol{e}_n = (0, 0, \ldots, 1)$, which shows that $\mathbb{R}^n$ is n dimensional. How about $\mathcal{F}$? Well, you can check that a basis consists of the functions $f(x_0) = 1$ as x_0 varies from $-\infty$ to ∞. There is an infinity of such functions, so $\mathcal{F}$ is infinite dimensional.

Vector spaces by themselves are not too interesting, so generally one adds a bit of structure to liven things up. The most important structure one can add to a vector space is called an inner product, which is just a generalization of the dot product. A vector space equipped with an inner product is called an **inner product space**. For the time being, we will restrict ourselves to the real field.

Definition. *An* **inner product** *on a vector space V over $\mathbb{R}$ is a map $(\cdot,\cdot) \to \mathbb{R}$ taking a pair of vectors to a real number, satisfying the following three properties:*

(1) $(u, av + bw) = a(u, v) + b(u, w)$ *(linearity on 2^{nd} entry)*

(2) $(u, v) = (v, u)$ *(symmetry)*

(3) $(u, u) \geq 0, = 0 \Leftrightarrow u = 0$ *(nondegeneracy).*

The classic example of an inner product is the usual dot product on $\mathbb{R}^n$, given by

$$(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i. \tag{1.8}$$

Let's check that it satisfies all the relevant properties. We have

$$\begin{aligned}(\boldsymbol{u}, a\boldsymbol{v} + b\boldsymbol{w}) &= \sum_{i=1}^{n} u_i(av_i + bw_i) = a\sum_{i=1}^{n} u_i v_i + b\sum_{i=1}^{n} u_i w_i \\ &= a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{w}),\end{aligned}$$

so property (1) holds. Next, we have

$$(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{n} u_i v_i = \sum_{i=1}^{n} v_i u_i = (\boldsymbol{v}, \boldsymbol{u}),$$

so property (2) holds. Finally, we have

$$(\boldsymbol{u}, \boldsymbol{u}) = \sum_{i=1}^{n} u_i^2 \geq 0,$$

because the square of a real number is always nonnegative. Furthermore, the expression on the right side above vanishes only if each u_i vanishes, but this just says that $\boldsymbol{u}$ is the zero vector. Hence the dot product is indeed an inner product.

How about an inner product on $\mathcal{F}$? Well, here we run into a difficulty, having to do with the fact that $\mathcal{F}$ is really just too large and unwieldy. So instead we define a smaller, more useful vector space $\mathrm{L}^2[a, b]$ consisting of all *square integrable* functions f on the interval $[a, b]$, which means that

$$\int_a^b [f(x)]^2\, dx < \infty. \tag{1.9}$$

Addition of two such functions, and multiplication by scalars is defined as for $\mathcal{F}$, but the square integrability condition allows us to define an inner product: [2]

$$(f, g) = \int_a^b f(x) g(x)\, dx. \tag{1.10}$$

You can check for yourself that this indeed satisfies all the properties of an inner product. [3]

With an inner product at our disposal, we can define many more properties of vectors. Two vectors u and v are **orthogonal** if $(u, v) = 0$, and the **length** of a vector u is $(u, u)^{1/2}$ (where the positive root is understood). Notice that, for $\mathbb{R}^n$, these are exactly the same definitions with which you are familiar. [4] The terminology is the same for any inner product space.

[2]Technically, we ought to write this as $(f, g)_{[a,b]}$ to remind ourselves that the integral is carried out for the interval $[a, b]$, but people are lazy and never do this. The interval will usually be clear from context.

[3]Actually, you will have to use the (generalized) Cauchy-Schwarz inequality, which says that $(f, g)^2 \leq (f, f)(g, g)$.

[4]Mathematicians write $\mathbb{E}^n$ for the vector space $\mathbb{R}^n$ equipped with the usual dot product, and they call it **Euclidean n-space**, to emphasize that this is the natural arena for Euclidean geometry. We will continue to be sloppy and use $\mathbb{R}^n$ for $\mathbb{E}^n$, which is bad form, but commonplace enough.

Thus, as $\mathrm{L}^2[a,b]$ is a space of functions, one says that the two functions f and g are orthogonal on the interval $[a,b]$ if $(f,g)=0$.

1.3 How to Invert a Fourier Series

Let us return now to where we left off. If

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

on the interval $[-\pi,\pi]$, how do we find the coefficients a_n and b_n? The answer is provided by exploiting a very important property of $\sin nx$ and $\cos nx$, namely that they each form a set of orthogonal functions on $[-\pi,\pi]$. That is, for $m,n\in\{0,1,2,\dots\}$,

$$(\sin mx,\sin nx)=\begin{cases}\pi\delta_{nm}, & \text{if } m,n\neq 0,\\ 0, & \text{if } m=n=0.\end{cases} \tag{1.11}$$

$$(\cos mx,\cos nx)=\begin{cases}\pi\delta_{nm}, & \text{if } m,n\neq 0,\\ 2\pi, & \text{if } m=n=0.\end{cases} \tag{1.12}$$

$$(\sin mx,\cos nx)=0, \tag{1.13}$$

where

$$(f,g)=\int_{-\pi}^{\pi}f(x)g(x)\,dx \tag{1.14}$$

is the inner product on the vector space of square integrable functions on $[-\pi,\pi]$. Equation (1.11), for example, says that $\sin nx$ and $\sin mx$ are orthogonal functions on $[-\pi,\pi]$ whenever n and m are different. They are not orthonormal, however, because $(\sin nx,\sin nx)=\pi\neq 1$. This is not a problem, but it does mean we must be mindful of factors of π everywhere.

Let's prove (1.11), as the other equations follow similarly. Recall the trigonometric identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{1.15}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha. \tag{1.16}$$

Adding and subtracting these equations gives

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right). \tag{1.17}$$

Hence,

$$\begin{aligned}(\sin nx, \sin mx) &= \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(n-m) - \cos(n+m))\, dx.\end{aligned} \tag{1.18}$$

Now we observe that

$$\int_{-\pi}^{\pi} \cos kx \, dx = 2\pi\delta_{k,0}, \tag{1.19}$$

because

$$\int_{-\pi}^{\pi} \cos kx \, dx = \begin{cases} \frac{1}{k}\sin kx\Big|_{-\pi}^{\pi} = 0, & \text{if } k \neq 0, \\ 2\pi, & \text{if } k = 0. \end{cases}$$

It follows from (1.18) and (1.19) and the fact that n and m are both positive integers, that

$$(\sin nx, \sin mx) = \frac{1}{2}(2\pi\delta_{n-m,0} - 2\pi\delta_{n+m,0}) = \pi\delta_{nm}. \tag{1.20}$$

Now, given Equation (1.2), the properties of the inner product together with Equations (1.11)-(1.13) allow us to compute:

$$\begin{aligned}(f, \sin mx) &= \frac{1}{2}\cancel{(a_0, \sin mx)} + \sum_{n=1}^{\infty} a_n \cancel{(\cos nx, \sin mx)} \\ &\quad + \sum_{n=1}^{\infty} b_n (\sin nx, \sin mx) \\ &= \sum_{n=1}^{\infty} b_n \cdot \pi \delta_{nm} = \pi b_m \quad \Rightarrow \quad b_m = \frac{1}{\pi}(f, \sin mx).\end{aligned}$$

A similar calculation for a_m yields the **inversion formulae for Fourier series**:

$$\boxed{\begin{aligned} a_n &= \frac{1}{\pi}(f, \cos nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n &= \frac{1}{\pi}(f, \sin nx) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned}} \tag{1.21}$$

Note that, because of the way we separated out the a_0 term in (1.2), the first inversion formula in (1.21) works for $n = 0$ as well.

1.4 Odd *vs.* Even Functions

We now wish to calculate some Fourier series, meaning, given a function $f(x)$, compute the Fourier coefficients a_n and b_n. Before we do this, though, we derive a few useful theorems regarding integrals of even and odd functions.

Definition. *A function $f(x)$ is* **even** *if $f(-x) = f(x)$, and* **odd** *if $f(-x) = -f(x)$.*

Graphically, an even function is symmetric about the vertical axis, while an odd function is antisymmetric about the vertical axis, as shown in Figures 2 and 3, respectively. Note that x^k is odd if k is odd, and even if k is

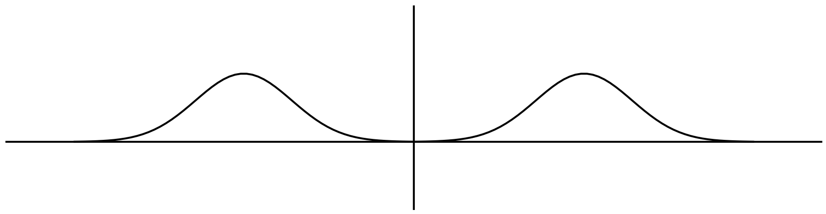

Figure 2: An even function

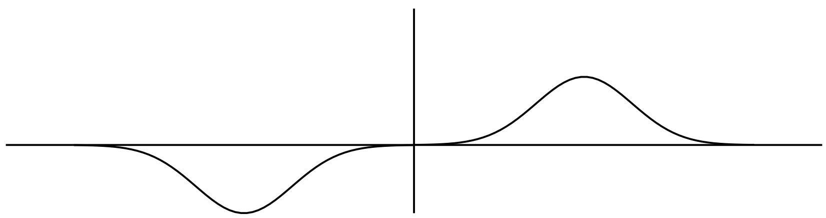

Figure 3: An odd function

even.

Theorem 1.1. *Let $g(x)$ be even. Then*

$$\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx. \tag{1.22}$$

Proof.

$$\begin{aligned}\int_{-a}^{a} g(x)\,dx &= \int_{-a}^{0} g(x)\,dx + \int_{0}^{a} g(x)\,dx \\ &= \int_{a}^{0} g(-x)\,d(-x) + \int_{0}^{a} g(x)\,dx \\ &= \int_{0}^{a} g(x)\,dx + \int_{0}^{a} g(x)\,dx \\ &= 2\int_{0}^{a} g(x)\,dx.\end{aligned}$$

□

Theorem 1.2. *The integral of an odd function over an even interval is zero.*

Proof. Let $f(x)$ be odd. Then

$$\int_{-a}^{a} f(x)\,dx = \int_{a}^{-a} f(-x)\,d(-x) = -\int_{-a}^{a} f(x)\,dx = 0.$$

□

Theorem 1.3. *Let $f(x)$ be odd and $g(x)$ be even. Then*

$$\int_{-a}^{a} f(x)g(x)\,dx = 0.$$

Proof. $f(x)g(x)$ is odd. □

1.5 Examples of Fourier Transforms

At last we come to our first example.

Example 1. $f(x) = x$ for $x \in [-\pi, \pi]$. By Theorem 1.3,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0,$$

because x is odd but $\cos nx$ is even, so all the a_n vanish. Next, we have

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx \\
&= \frac{2}{\pi}\left(-\frac{d}{dk}\int_{0}^{\pi} \cos kx \, dx\right)\bigg|_{k=n} \\
&= -\frac{2}{\pi}\cdot\frac{d}{dk}\left(\frac{1}{k}\sin kx\right)\bigg|_{k=n} \\
&= \frac{2}{\pi}\left(\frac{1}{k^2}\sin k\pi - \frac{\pi}{k}\cos kx\right)_{k=n} \\
&= \boxed{\frac{2}{n}(-1)^{n+1}}.
\end{aligned}$$

Hence, on the interval $[-\pi, \pi]$

$$\boxed{x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx}. \tag{1.23}$$

(This is certainly a complicated way of writing x (!))

Example 2.

$$f(x) = \begin{cases} 1, & \text{if } 0 < x \le \pi, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } -\pi \le x < 0. \end{cases} \tag{1.24}$$

Again $a_n = 0$ for all n. Also,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \sin nx\,dx = -\frac{2}{n\pi}\cos nx\bigg|_0^\pi \\ &= -\frac{2}{n\pi}\left((-1)^n - 1\right) \\ &= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{4}{n\pi}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{1.25}$$

Hence

$$\boxed{f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}}. \tag{1.26}$$

The right hand side of (1.26) is an infinite sum. In Figure 4 we have illustrated the shape of the function that results when one takes only the first term, the first three terms, and the first eleven terms. It turns out that, no matter how many terms we use, the sum on the right hand side always has some wiggles left in it, so that it fails to exactly match the step function (1.24) (also shown in Figure 4). This phenomenon is called **ringing**, or the **Gibbs phenomenon**, after the physicist who first analyzed it in detail. It occurs because the function $f(x)$ is discontinuous at the origin. Thus, in some sense (1.26) is a lie, because it is not an exact equality. When we wish to emphasize this fact, we will write

$$\boxed{F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)}, \tag{1.27}$$

and call this the Fourier series representation of $f(x)$. It then make sense to ask under what conditions $F(x) = f(x)$ holds. This is precisely the question of convergence, to which we now turn.

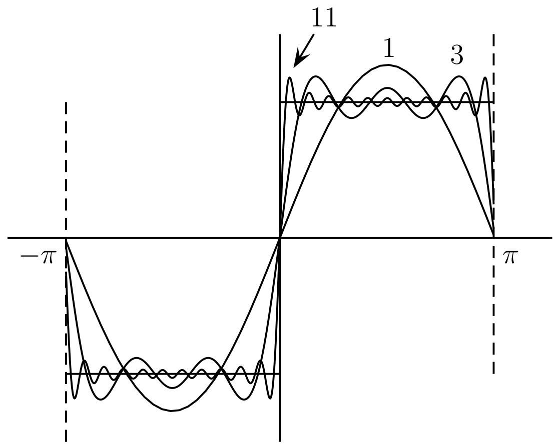

Figure 4: The Gibbs phenomenon

1.6 Convergence in the Mean

How bad can this ringing phenomenon be? That is, when does the Fourier series $F(x)$ converge to the original function $f(x)$? This is a subtle and complicated question with which mathematicians have struggled for over a century. Fortunately, for our purposes, there is a very natural answer for the functions most likely to arise in practice, namely the **piecewise continuous** functions.

Because the Fourier series $F(x)$ may not converge at all, we must proceed indirectly. Let $\{g_n\}$ be the sequence of partial sums of the Fourier series:

$$g_n(x) := \frac{A_0}{2} + \sum_{m=1}^{n} (A_m \cos mx + B_m \sin mx). \tag{1.28}$$

(Obviously, $F(x) = \lim_{n\to\infty} g_n(x)$, provided the limit exists.) Define

$$D_n := \int_{-\pi}^{\pi} [f(x) - g_n(x)]^2 \, dx. \tag{1.29}$$

D_n is called the **total square deviation** of $f(x)$ and $g_n(x)$.

Theorem 1.4. *The total square deviation D_n is minimized if the constants A_n and B_n in* (1.28) *are chosen to be the usual Fourier series coefficients.*

Proof. Recall the definition of the inner product

$$(f, g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

We wish to minimize

$$D_n = (f - g_n, f - g_n).$$

Using the symmetry and bilinearity of the inner product together with the orthogonality relations (1.11)-(1.13) we find

$$\begin{aligned}
D_n = \Bigg(& f - \frac{A_0}{2} - \sum_{m=1}^{n}(A_m \cos mx + B_m \sin mx)\,, \\
& f - \frac{A_0}{2} - \sum_{m=1}^{n}(A_m \cos mx + B_m \sin mx)\Bigg) \\
= {} & (f, f) - (f, A_0) + \frac{\pi}{2}A_0^2 \\
& - 2\sum_{m=1}^{n}(A_m(f, \cos mx) + B_m(f, \sin mx)) \\
& + \pi\sum_{m=1}^{n}(A_m^2 + B_m^2)
\end{aligned}$$

Let $k \neq 0$. Treating the coefficients as independent variables, we get

$$0 = \frac{\partial D_n}{\partial A_k} = 2\pi - 2(f, \cos kx),$$

which implies

$$A_k = \frac{1}{\pi}(f, \cos kx) = a_k.$$

Similar analysis shows that $A_0 = a_0$ and $B_k = b_k$. Thus the total square deviation is minimized if the coefficients are chosen to be the usual Fourier coefficients, as claimed. □

To find the minimum value of D_n, we substitute the Fourier coefficients into D_n to get

$$D_{n,min} = (f, f) - \pi(\tfrac{1}{2}a_0^2 + \sum_{m=1}^{n}(a_m^2 + b_m^2)).$$

Since D_n is nonnegative, we get the following inequality for all n:

$$\frac{1}{2}a_0^2 + \sum_{m=1}^{n}(a_m^2 + b_m^2) \leq \frac{1}{\pi}(f, f).$$

Now take the limit as $n \to \infty$. The sequence on the left is bounded by the quantity on the right, and is monotone nondecreasing. So it possesses a limit, which satisfies the inequality

$$\boxed{\frac{1}{2}a_0^2 + \sum_{m=1}^{\infty}(a_m^2 + b_m^2) \leq \frac{1}{\pi}(f, f)}. \tag{1.30}$$

This result is known as **Bessel's inequality**.

Next we introduce another definition.

Definition. *A sequence of functions* $\{g_n(x)\}$ *is said to* **converge in the mean** *to a function* $f(x)$ *on the interval* $[a, b]$ *if*

$$\lim_{n\to\infty} \int_a^b [f(x) - g_n(x)]^2\, dx = 0. \tag{1.31}$$

If we could show that equality holds in Bessel's inequality, this would show that $\{g_n(x)\}$ converges in the mean to $f(x)$. In that case, we would say that the Fourier series $F(x)$ converges in the mean to $f(x)$. (This is what is meant by the equal sign in (1.2)). Fortunately, it is true for a large class of functions.

Theorem 1.5. *Let $f(x)$ be piecewise continuous. Then the Fourier series $F(x)$ converges in the mean to $f(x)$.*

For a proof of this theorem, consult a good book on Fourier series.

1.7 Parseval's Identity

From now on we will suppose that $f(x)$ is piecewise continuous. In that case, Bessel's inequality becomes an equality, usually called **Parseval's identity**:

$$\boxed{\frac{1}{2}a_0^2 + \sum_{m=1}^{\infty}(a_m^2 + b_m^2) = \frac{1}{\pi}(f, f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx}. \tag{1.32}$$

Parseval's identity can be used to prove various interesting identities. For example, for the step function

$$f(x) = \begin{cases} 1, & \text{if } 0 < x \leq \pi, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } -\pi \leq x < 0. \end{cases}$$

which is piecewise continuous, we computed the Fourier coefficients in (1.25): $a_n = 0$ and

$$b_n = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \dfrac{4}{n\pi}, & \text{if } n \text{ is odd}. \end{cases}$$

The left hand side of Parseval's identity becomes

$$\sum_{n=1}^{\infty} b_n^2 = \frac{16}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2},$$

whereas the right hand side becomes

$$\frac{1}{\pi}(f, f) = \frac{1}{\pi} \int_{-\pi}^{\pi} dx = 2.$$

Equating the two gives the amusing result that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}. \tag{1.33}$$

1.8 Periodic Extensions

The Fourier series $F(x)$ of a function $f(x)$ defined on the interval $[-\pi, \pi]$ is not itself restricted to that interval. In particular, if

$$F(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then

$$\boxed{F(x + 2\pi) = F(x)}. \tag{1.34}$$

We say that the Fourier series representation $F(x)$ of $f(x)$ is the **periodic extension** of $f(x)$ to the whole line. What this actually means is that $F(x)$ converges in the mean to the periodic extension of $f(x)$.

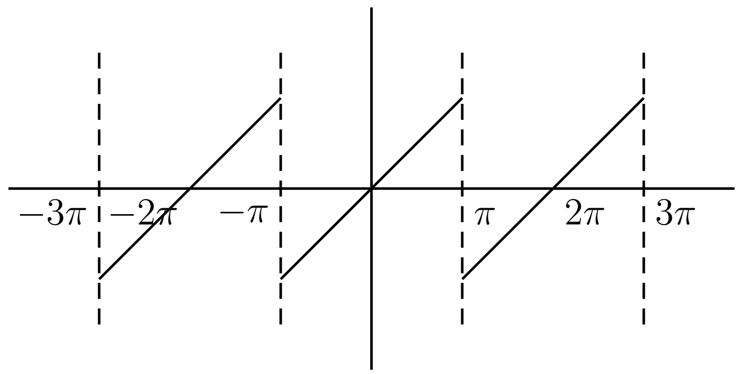

Figure 5: The sawtooth function as a periodic extension of a line segment

For example, if

$$f(x) = \begin{cases} x, & \text{for } -\pi < x \leq \pi, \\ 0, & \text{otherwise.} \end{cases}$$

then $F(x)$ is (more precisely, converges in the mean to) the sawtooth function illustrated in Figure 5.

1.9 Arbitrary Intervals

There is no reason to restrict ourselves artificially to functions defined on the interval $[-\pi, \pi]$. The entire development easily extends to functions defined on an arbitrary finite interval $[-L, L]$ instead of $[-\pi, \pi]$. The simple change of variable

$$\boxed{x \to \frac{\pi}{L}x} \tag{1.35}$$

gives

$$\boxed{\begin{aligned} F(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\frac{n\pi x}{L}\,dx \end{aligned}} \tag{1.36}$$

In this case,

$$\boxed{F(x+2L) = F(x)}, \tag{1.37}$$

showing that the Fourier series $F(x)$ is a periodic extension of $f(x)$, now with period $2L$. *From now on we will follow the standard convention and not distinguish between $F(x)$ and $f(x)$, but you should always understand the sense in which $F(x)$ is really just an approximation to (the periodic extension of) $f(x)$.*

1.10 Complex Fourier Series

The connection between a function and its Fourier series expansion can be written more compactly by appealing to complex numbers. Begin by using Euler's formula to write

$$\cos\frac{n\pi x}{L} = \frac{1}{2}\left[e^{in\pi x/L} + e^{-in\pi x/L}\right] \tag{1.38}$$

$$\sin\frac{n\pi x}{L} = \frac{1}{2i}\left[e^{in\pi x/L} - e^{-in\pi x/L}\right]. \tag{1.39}$$

Substitute these expressions into the first equation in (1.36) to get

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - ib_n)e^{in\pi x/L} + \frac{1}{2}(a_n + ib_n)e^{-in\pi x/L} \right] \\
&= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n)e^{in\pi x/L} + \sum_{n=-1}^{-\infty} \frac{1}{2}(a_{-n} + ib_{-n})e^{in\pi x/L} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\pi x/L} + \sum_{n=-1}^{-\infty} c_n e^{in\pi x/L} \\
&= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L},
\end{aligned}
$$

where

$$
c_n = \begin{cases} \frac{1}{2}(a_n - ib_n), & \text{if } n > 0, \\ \dfrac{a_0}{2}, & \text{if } n = 0, \\ \frac{1}{2}(a_{-n} + ib_{-n}), & \text{if } n < 0. \end{cases}
$$

Next, use the inversion formulae in (1.36) to compute the complex coefficients c_n in terms of the original function $f(x)$. For example, if $n > 0$:

$$
\begin{aligned}
c_n &= \frac{1}{2}(a_n - ib_n) \\
&= \frac{1}{2L} \int_{-L}^{L} f(x) \left(\cos \frac{n\pi x}{L} - i \sin \frac{n\pi x}{L} \right) dx \\
&= \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} \, dx.
\end{aligned}
$$

A similar computation shows that this formula remains valid for all the other values of n. We therefore obtain the **complex Fourier series** representation

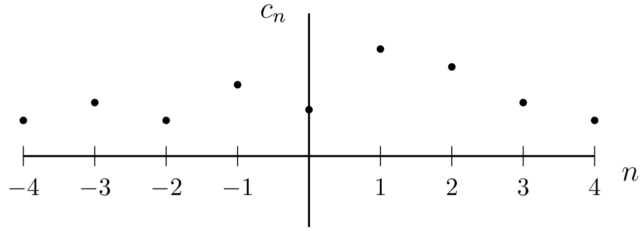

Figure 6: A discrete Fourier spectrum

of a function $f(x)$ defined on an interval $[-L, L]$,

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}}, \tag{1.40}$$

together with the inversion formula

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L}\, dx}. \tag{1.41}$$

2 The Fourier Transform

2.1 The Limit of a Fourier Series

A natural question to ask at this stage is what happens when $L \to \infty$? That is, can we represent an arbitrary function on the real line as some kind of sum of simpler functions? The answer leads directly to the idea of the Fourier transform.

We may interpret (1.40) as saying that the function $f(x)$ on the interval $[-L, L]$ is determined by its (discrete) **Fourier spectrum**, namely the set of all complex constants $\{c_n\}$. We can represent this metaphorically by means

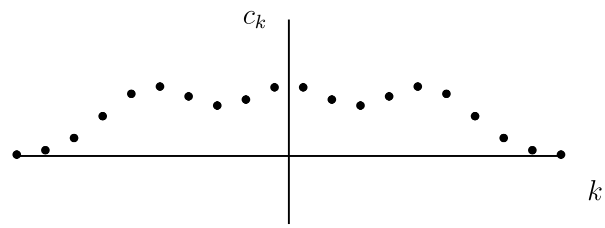

Figure 7: A discrete Fourier spectrum in terms of wavenumbers

of the graph in Figure 6. [5]

It is useful to introduce a new variable

$$\boxed{k := \frac{n\pi}{L}}. \tag{2.1}$$

k is called the **spatial frequency** or **wavenumber** associated to the **mode** n. In terms of k, the Fourier spectrum is now given by coefficients depending on k, as in Figure 7.

Now consider the Fourier "series" of a **nonperiodic** function $f(x)$. This means that, in order to capture the entire function, we must take

$$L \to \infty. \tag{2.2}$$

In this limit the spatial frequencies become **closely spaced** and the discrete **Fourier spectrum** becomes a **continuous** Fourier spectrum, as in Figure 8. This continuous Fourier spectrum is precisely the **Fourier transform** of $f(x)$.

To explore the limit (2.2), begin with Equations (1.40) and (1.41) and

[5]Technically, this graph makes no sense, because c_n is a complex number, but we are treating it here as if it were real—hence the word 'metaphorically'.

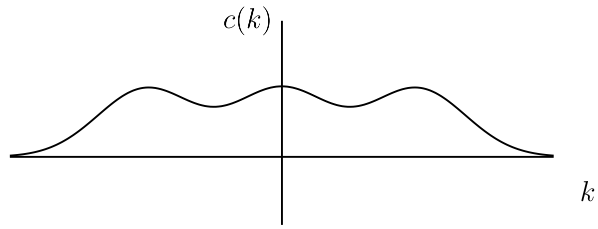

Figure 8: A continuous Fourier spectrum

use the definition (2.1):

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \Delta n \nearrow 1 \\ &= \sum_{kL/\pi=-\infty}^{\infty} c_{kL/\pi} e^{ikx} \frac{L}{\pi} \Delta k \\ &= \sum_{kL/\pi=-\infty}^{\infty} c_L(k) e^{ikx} \Delta k, \end{aligned}$$

where

$$c_L(k) := \frac{L}{\pi} c_{kL/\pi} = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{-ikx}\, dx.$$

Now take the limit $L \to \infty$. Then $c_L(k) \to c(k)$, where

$$c(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx,$$

and

$$f(x) = \int_{-\infty}^{\infty} c(k) e^{ikx}\, dk.$$

Lastly, to make things look more symmetric, we change the normalization of the spectrum coefficients, by defining

$$\widetilde{f}(k) := \sqrt{2\pi} c(k).$$

The final result is

$$\boxed{\widetilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx}, \tag{2.3}$$

called the **Fourier spectrum** or **Fourier amplitudes** or **Fourier transform of** $f(x)$, and

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\, dk}, \tag{2.4}$$

called the **inverse Fourier transform of** $\widetilde{f}(k)$. If $f(x)$ is periodic on a finite interval $[-L, L]$, these expressions reduce back to (1.41) and (1.40), respectively. But they hold more generally for any function. [6]

Equation (2.4) reveals how any function $f(x)$ can be expanded as a sum (really, integral) of elementary harmonic functions e^{ikx} weighted by different amplitudes $\widetilde{f}(k)$. Moreover, the similarity between (2.3) and (2.4) shows that, in essence, $f(x)$ and $\widetilde{f}(k)$ contain the same information. Given one, you can obtain the other.

In (2.3) and (2.4), the variables x and k are naturally paired. Typically, one interprets e^{ikx} as a wave, in which case x is the distance along the wave and k becomes the usual wavenumber $k = 2\pi/\lambda$. But often one uses t instead of x, and ω instead of k, where t denotes 'time' and ω denotes 'angular frequency', in which case the Fourier transform and its inverse become

[6] Well, not quite. The analogue of the question of convergence for Fourier series now becomes a question of the existence of the integrals, which we will conveniently ignore.

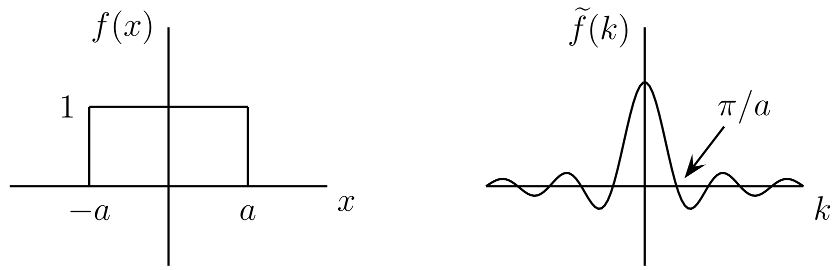

Figure 9: The box function and its Fourier transform

$$\widetilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt \tag{2.5}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(\omega) e^{i\omega t}\, d\omega \tag{2.6}$$

The pairs (x, k) and (t, ω) are referred to as **conjugate variables**. In either case, the function being transformed is decomposed as a sum of harmonic waves.

2.2 Examples

Example 3. [**The box function** or **square filter**] (See Figure 9.)

$$f(x) = \begin{cases} 1, & \text{if } |x| < a, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

$$
\begin{aligned}
\widetilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ika} - e^{ika}}{-ik} \\
&= \frac{2a}{\sqrt{2\pi}} \frac{\sin ka}{ka} = \boxed{\frac{2a}{\sqrt{2\pi}} \operatorname{sinc}(ka)}. \qquad (2.8)
\end{aligned}
$$

(This last equality defines the 'sinc' function. It appears often enough that it is given a name.)

Example 4. [The Gaussian] The Gaussian function is one of the most important functions in mathematics and physics. Here we derive an interesting and useful property of Gaussians.

Begin with the Gaussian function

$$
f(x) = Ne^{-\alpha x^2}, \qquad (2.9)
$$

where N is some normalization constant whose exact value is unimportant right now. We wish to compute its Fourier transform, namely

$$
\widetilde{f}(k) = \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\alpha x^2} e^{-ikx}\, dx.
$$

The trick is to complete the square in the exponent:

$$
\begin{aligned}
-\alpha x^2 - ikx &= -\alpha \left(x^2 + \frac{ik}{\alpha} x \right) \\
&= -\alpha \left(x^2 + \frac{ik}{\alpha} x + \left(\frac{ik}{2\alpha} \right)^2 - \left(\frac{ik}{2\alpha} \right)^2 \right) \\
&= -\alpha \left(x + \frac{ik}{2\alpha} \right)^2 - \frac{k^2}{4\alpha}.
\end{aligned}
$$

Next, change variables from x to u:

$$u^2 = \alpha\left(x + \frac{ik}{2\alpha}\right)^2 \quad \Rightarrow \quad u = \sqrt{\alpha}\left(x + \frac{ik}{2\alpha}\right) \quad \Rightarrow \quad du = \sqrt{\alpha}\, dx.$$

This gives

$$\begin{aligned}\widetilde{f}(k) &= \frac{1}{\sqrt{2\pi}}\frac{N}{\sqrt{\alpha}}\int_{-\infty+i\beta}^{\infty+i\beta} e^{-u^2} e^{-k^2/4\alpha}\, du \\ &= \frac{N}{\sqrt{2\pi\alpha}} e^{-k^2/4\alpha}\int_{-\infty}^{\infty} e^{-u^2}\, du,\end{aligned}$$

where $\beta := k\sqrt{\alpha}/2$. Fortunately, we can simply drop the β terms in the limits. (This follows from Cauchy's theorem in complex analysis.)

We have reduced the problem to computing a standard Gaussian integral. As the value of this integral pops up in many places, we illustrate the trick used to obtain its value.

Lemma 2.1.

$$\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}. \tag{2.10}$$

Proof. Define

$$I := \int_{-\infty}^{\infty} e^{-u^2}\, du.$$

Then

$$\begin{aligned}I^2 &= \int_{-\infty}^{\infty} e^{-u^2}\, du \int_{-\infty}^{\infty} e^{-v^2}\, dv = \int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, du\, dv \\ &= \int_0^{\infty} dr \int_0^{2\pi} e^{-r^2}\, r\, dr\, d\theta = -\pi e^{-r^2}\Big|_0^{\infty} = \pi.\end{aligned}$$

Hence $I = \sqrt{\pi}$. □

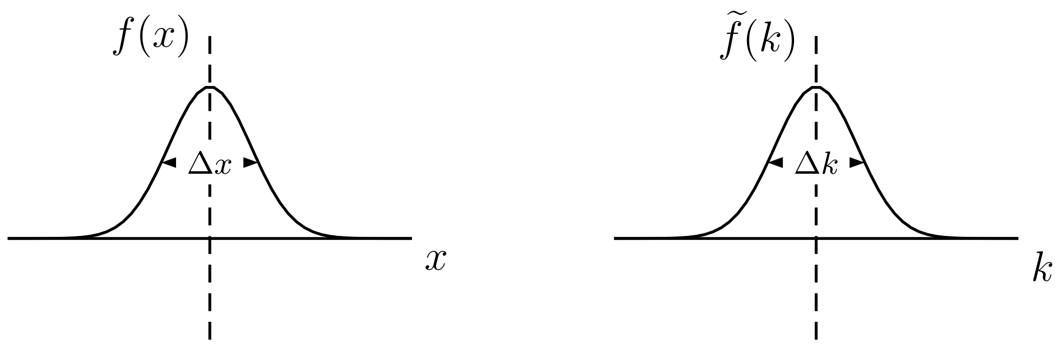

Figure 10: The Fourier transform of a Gaussian is a Gaussian

Returning to the main thread and putting everything together gives

$$\boxed{\widetilde{f}(k) = \frac{N}{\sqrt{2\alpha}} e^{-k^2/4\alpha}}. \tag{2.11}$$

In other words, **the Fourier transform of a Gaussian is a Gaussian** (See Figure 10.)

Observe that the width Δx of $f(x)$ is proportional to $1/\sqrt{\alpha}$, while the width Δk of $\widetilde{f}(k)$ is proportional to $\sqrt{\alpha}$. Hence, as one function gets narrower, the other gets wider. In passing we remark that this fact is intimately related to **Heisenberg Uncertainty Principle**. In quantum theory the wavefunction of a localized particle may be described by a wavepacket, and its momentum space wavefunction is essentially its Fourier transform. If the wavepacket is Gaussian in shape, then so is its momentum space wavefunction. As a consequence, the more localized the particle, the less well known is its momentum.

3 The Dirac Delta Function

Equations (2.3) and (2.4) are supposed to be inverses of one another. That is, given $f(x)$, one can use (2.3) to find $\widetilde{f}(k)$, so plugging $\widetilde{f}(k)$ into (2.4) one should recover the original function $f(x)$. It turns out that, as with the case of Fourier series, this is not generally true. However, we will restrict ourselves to those functions for which it is true, namely the class of *sufficiently smooth test functions.*

Restricting ourselves to this class, we must have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{f}(k) e^{ikx}\, dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} dx'\, f(x') e^{-ikx'} \\ &= \int_{-\infty}^{\infty} dx'\, f(x') \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \right\} \\ &= \int_{-\infty}^{\infty} dx'\, f(x') \delta(x - x'), \end{aligned} \tag{3.1}$$

where we define

$$\boxed{\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}}. \tag{3.2}$$

$\delta(x - x')$ is called a **Dirac delta function**, and it takes a bit of getting used to. By virtue of (3.1) it must satisfy the following properties:

1. The most important property (indeed, its defining property) is just (3.1) itself, which is called the **sifting property**.

$$\boxed{\int_{-\infty}^{\infty} f(x') \delta(x - x')\, dx' = f(x)} \qquad \text{(sifting property).} \tag{3.3}$$

It is called the 'sifting property' for the simple reason that, when integrated, the delta function selects, or sifts out, the value at $x' = x$ of the function against which it is integrated. In other words, it acts like a sort of continuous analogue of the Kronecker delta.

2. The delta function is an even function:

$$\delta(x - x') = \delta(x' - x). \tag{3.4}$$

This follows from (3.2) by changing variables from k to $-k$:

$$\begin{aligned}\delta(x' - x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x'-x)} = -\frac{1}{2\pi}\int_{\infty}^{-\infty} dk\, e^{-ik(x'-x)} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ik(x'-x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x')} \\ &= \delta(x - x').\end{aligned}$$

3. The sifting property is often written as follows. For any sufficiently smooth test function f,

$$\boxed{\int_{-\infty}^{\infty} f(z)\delta(z)\, dz = f(0)}. \tag{3.5}$$

This is entirely equivalent to (3.3), as we now show. It is convenient here to rewrite (3.3) using Property 2 as

$$\int_{-\infty}^{\infty} f(x')\delta(x' - x)\, dx' = f(x). \tag{3.6}$$

Setting $x = 0$ in (3.6) yields (3.5) immediately. On the other hand, if we define a new function $g(z) := f(z - x)$, so that $f(z) = g(z + x)$, then (3.5) gives

$$\int_{-\infty}^{\infty} g(z+x)\delta(z)\,dz = g(x).$$

Now change variables from z to $x' = z + x$ to get

$$\int_{-\infty}^{\infty} g(x')\delta(x'-x)\,dx' = g(x),$$

which is (3.6).

4. Look at (3.5) closely. On the left side we are adding up various values of the function $f(z)$ times the delta function $\delta(z)$, while on the right side we have only one value of the function, namely $f(0)$. This means that the delta function must vanish whenever $z \neq 0$ (otherwise we would get an equation like $f(0) + \text{other stuff} = f(0)$, which is absurd). Thus, we must have,

$$z \neq 0 \Rightarrow \delta(z) = 0. \tag{3.7}$$

5. Now let f be the unit function, namely the function $f(z) = 1$ for all z. Plugging this into (3.5) gives

$$\int_{-\infty}^{\infty} \delta(z)\,dz = 1. \tag{3.8}$$

In other words, the delta function has *unit area.*

6. By Property 4, the delta function vanishes whenever $z \neq 0$, which means that it is nonzero only when $z = 0$. That is, only $\delta(0)$ is nonzero. What is the value of $\delta(0)$? By Property 5, the delta function has unit area, which means that $\delta(0)$ must be *infinite*! (The reason is that dx is an infinitessimal, and if $\delta(x)$ were a finite value at the single point $x = 0$, the integral would be zero, not unity.)

All these properties add up to one strange object. It is manifestly **not a function**, for the simple reason that infinity is not a well-defined value. Instead, it is called a **generalized function** or **distribution**. In an effort to restore a modicum of rigor to the entire enterprise, mathematicians define the Dirac delta function as follows.

Definition. *A sequence* $\{\phi_n\}$ *of functions is called a* **delta sequence** *if*

$$\boxed{\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n(x)f(x)\,dx = f(0)} \tag{3.9}$$

for all sufficiently smooth test functions $f(x)$*.*

In other words, $\{\phi_n\}$ is a delta sequence if it obeys the sifting property in the limit as $n \to \infty$. One then formally defines the Dirac delta function to be the limit of a delta sequence,

$$\boxed{\delta(x) = \lim_{n\to\infty}\phi_n(x)}, \tag{3.10}$$

even though this limit **does not exist**! There are many different possible delta sequences, but in some sense they all have the same limit.

3.1 Examples of Delta Sequences

There is a lot of freedom in the definition of a delta sequence, but (3.9) does impose a few restrictions on the $\{\phi_n\}$. For example, setting f to be the identity function again gives the requirement that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi_n(x)\,dx = 1. \tag{3.11}$$

That is, in the limit, the integrals must be normalized to unity.

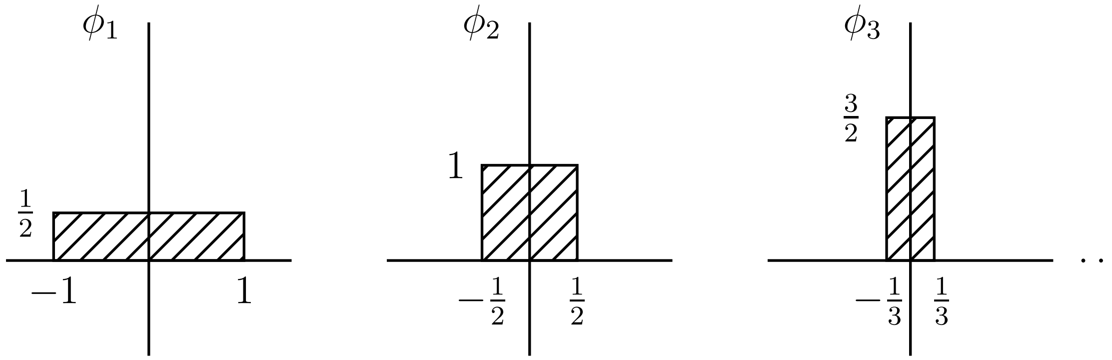

Figure 11: The delta sequence (3.12). Note that all graphs have unit area.

We now offer some examples of delta sequences.

1.

$$\phi_n(x) = \begin{cases} 0, & \text{if } |x| \geq 1/n, \\ n/2, & \text{if } |x| < 1/n. \end{cases} \tag{3.12}$$

This is illustrated in Figure 11.

2.

$$\phi_n(x) = \frac{n}{\pi} \, \frac{1}{1 + n^2 x^2}. \tag{3.13}$$

3.

$$\phi_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}. \tag{3.14}$$

4.

$$\phi_n(x) = \frac{1}{n\pi} \frac{\sin^2 nx}{x^2}. \tag{3.15}$$

Let's show that (3.12) is a delta sequence. First, we recall the **mean value theorem** from calculus.

Theorem 3.1. *Let f be a continuous real function on an interval $[a, b]$. Then* *there exists a point* $\xi \in [a, b]$ *such that*

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(\xi).$$

Applying this theorem, we have

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_n(x) f(x)\,dx &= \int_{-1/n}^{1/n} \left(\frac{n}{2}\right) f(x)\,dx \\ &= \left(\frac{n}{2}\right)\left(\frac{2}{n}\right) f(\xi) \qquad \text{(mean value theorem)} \\ &= f(\xi).\end{aligned}$$

As $-1/n \le \xi \le 1/n$, $n \to \infty$ forces $\xi \to 0$. The **continuity** of $f(x)$ then gives $f(\xi) \to f(0)$. □

3.2 Delta Calculus

Although the delta function is not really a function, we can write down some relations, called **delta function identities**, that sometimes come in handy in calculations. Taken by themselves, the following relations make no sense, unless they are viewed as holding whenever both sides are integrated up against a sufficiently smooth test function.

1.

$$\boxed{x\delta(x) = 0}. \tag{3.16}$$

This holds because

$$\int x f(x)\delta(x)\,dx = \lim_{x\to 0} x f(x) = 0 \qquad \text{if } |f| < \infty.$$

2.

$$\boxed{\delta(ax) = \frac{1}{|a|}\delta(x) \quad \text{for } a \neq 0}. \tag{3.17}$$

Assume $a > 0$. Let $\xi = ax$ and $d\xi = a\,dx$. Then

$$\int_{-\infty}^{\infty} \delta(ax) f(x)\,dx = \int_{-\infty}^{\infty} \delta(\xi) f(\xi/a)\,d\xi/a = \frac{1}{a} f(0).$$

The other case is similar.

3.

$$\boxed{\delta(x) = \delta(-x)} \qquad (\delta \text{ is even}). \tag{3.18}$$

Set $a = -1$ in (3.17). (Of course, we already know this, by virtue of (3.4).)

4.

$$\boxed{\delta(g(x)) = \sum_{\substack{\text{simple zeros}\\ \text{of } g(x)}} \frac{\delta(x - x_i)}{|g'(x_i)|}}. \tag{3.19}$$

(x_i is a **simple zero** of $g(x)$ if $g(x_i) = 0$ but $g'(x_i) \neq 0$.) Write $g(x) \approx g'(x_i)(x - x_i)$ near a simple zero x_i, then use (3.17):

$$\delta[g(x)] = \frac{\delta(x - x_i)}{|g'(x_i)|}$$

If g has N simple zeros then, as $\delta[g(x)]$ is nonzero only near a zero of $g(x)$, we get

$$\begin{aligned}\int f(x)\delta[g(x)]\,dx &= \sum_{i=1}^{N}\int_{\text{near } x_i} f(x)\delta[g(x)]\,dx \\ &= \sum_{i=1}^{N}\int_{\text{near } x_i} f(x)\frac{\delta(x-x_i)}{|g'(x_i)|}\,dx \\ &= \sum_{i=1}^{N}\frac{f(x_i)}{|g'(x_i)|}.\end{aligned}$$

(Note that none of this makes sense if $g(x)$ has non-simple zeros, because then we would be dividing by zero.) The next example provides an amusing application of (3.19).

Example 5.

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2}\delta(\sin x)\,dx &= \int_{-\infty}^{\infty} e^{-x^2}\left(\sum_{n=-\infty}^{\infty}\frac{\delta(x-n\pi)}{|\cos n\pi|}\right)dx \\ &= \sum_{n=-\infty}^{\infty} e^{-(n\pi)^2}. \end{aligned} \tag{3.20}$$

5. **Derivatives of delta functions.** The derivative of a delta function is defined via integration by parts:

$$\boxed{\int_{-\infty}^{\infty}\delta'(x)f(x)\,dx = -f'(0)}. \tag{3.21}$$

Similar reasoning gives higher order derivatives of delta functions.

3.3 The Dirac Comb or Shah Function

A **Dirac comb** or **Shah function** Ш(t) is an infinite sum of Dirac delta functions

$$Ш_T(t) := \sum_{j=-\infty}^{\infty} \delta(t - jT), \tag{3.22}$$

By construction, it is periodic with period T. Hence it can be expanded in a Fourier series:

$$Ш_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t/T},$$

where

$$\begin{aligned} c_n &= \frac{1}{T}\int_{t_0}^{t_0+T} Ш_T(t) e^{-2\pi i n t/T}\,dt \\ &= \frac{1}{T}\sum_{j=-\infty}^{\infty}\int_{t_0}^{t_0+T} \delta(t - jT) e^{-2\pi i n t/T}\,dt \\ &= \frac{1}{T}. \end{aligned}$$

(The result is the same if $t_0 = 0$, but the computation is a bit more subtle, because one gets $1/2$ at each end.) Plugging back in gives

$$\boxed{Ш_T(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{2\pi i n t/T}}. \tag{3.23}$$

3.4 Three Dimensional Delta Functions

Extending the Dirac delta function to higher dimensions can be a bit tricky. It is no problem to define a higher dimensional analogue using Cartesian coordinates. For example, in three dimensions, we just have

$$\boxed{\delta(\boldsymbol{r}-\boldsymbol{r}')=\delta(x-x')\delta(y-y')\delta(z-z')}. \tag{3.24}$$

This satisfies the corresponding sifting property:

$$\int f(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{r}')\,d^3x=f(\boldsymbol{r}'), \tag{3.25}$$

which we again take to be the definition of the delta function. In spherical polar coordinates we demand that the following equation obtain:

$$\int f(r,\theta,\phi)\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')=f(r',\theta',\phi'). \tag{3.26}$$

The only way this can happen is if

$$\boxed{\delta(\boldsymbol{r}-\boldsymbol{r}')=\frac{1}{|r^2\sin\theta|}\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}, \tag{3.27}$$

because we must compensate for the Jacobian factor in (3.25) when we change variables. Notice the absolute value signs, which are required.

The three dimensional delta function in spherical polar coordinates actually arises in physics, especially in electrodynamics. If $\boldsymbol{\nabla}^2$ is the usual Laplacian operator, then the following result holds.

Theorem 3.2.

$$\boldsymbol{\nabla}^2\frac{1}{r}=-4\pi\delta(\boldsymbol{r}). \tag{3.28}$$

Proof. In spherical polar coordinates we found

$$\boldsymbol{\nabla}^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

It follows that, when $r \neq 0$, we get

$$\boldsymbol{\nabla}^2 \frac{1}{r} = 0.$$

What about $r = 0$? In that case, the Laplacian operator *itself* becomes infinite, so we get infinity. This looks awfully familiar. We have a function that is zero everywhere except at one point, where it is infinity. The only thing we need to check to make sure it is a delta function is that it has unit area.

To do this, recall that the Laplacian of a scalar function is just $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$ acting on that scalar function. The gradient operator in spherical polar coordinates is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}} \partial_r + \hat{\boldsymbol{\theta}} \frac{1}{r} \partial_\theta + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \partial_\phi,$$

so

$$\boldsymbol{\nabla} \frac{1}{r} = -\frac{\hat{\boldsymbol{r}}}{r^2}. \tag{3.29}$$

Now let V be a spherical volume. From Gauss' theorem we have

$$\int_V \boldsymbol{\nabla}^2 \frac{1}{r}\, d\tau = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \frac{1}{r}\, d\tau = -\oint_{\partial V} \frac{\hat{\boldsymbol{r}}}{r^2} \cdot d\boldsymbol{S} = -4\pi.$$

Thus $\boldsymbol{\nabla}^2(1/r)$ vanishes everywhere except at $r = 0$ where it is infinite, but its integral over a spherical volume is -4π. The only way this can happen is if $(1/4\pi)\boldsymbol{\nabla}^2(1/r)$ is a delta function. □

4 The Fourier Convolution Theorem

Next we turn to a very useful theorem relating the Fourier transform of a product of two functions to the Fourier transform of the individual functions.

Define the **convolution** of two functions f and g to be

$$\boxed{(f \star g)(x) := \int_{-\infty}^{\infty} d\xi\, f(\xi) g(x-\xi)}. \tag{4.1}$$

Although it does not look like it, this definition is actually symmetric.

Lemma 4.1.

$$f \star g = g \star f. \tag{4.2}$$

Proof. Let $\eta = x - \xi$. Then

$$\begin{aligned}(f \star g)(x) &= \int_{-\infty}^{\infty} d\xi\, f(\xi) g(x-\xi) \\ &= -\int_{+\infty}^{-\infty} d\eta\, f(x-\eta) g(\eta) \\ &= (g \star f)(x).\end{aligned}$$

□

The reason for introducing the notion of convolution is the following result.

Theorem 4.2 (Convolution Theorem for Fourier Transforms)**.**

$$\boxed{\widetilde{fg} = \frac{1}{\sqrt{2\pi}} \widetilde{f} \star \widetilde{g}}. \tag{4.3}$$

We say that the Fourier transform of the product is the convolution of the Fourier transforms (up to some irritating factors of $\sqrt{2\pi}$).

Proof.

$$
\begin{aligned}
\widetilde{(fg)}(k) &= \frac{1}{\sqrt{2\pi}} \int f(x) g(x) e^{-ikx}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int dx\, e^{-ikx}\, f(x) \left[\frac{1}{\sqrt{2\pi}} \int \widetilde{g}(k')\, e^{ik'x}\, dk' \right] \\
&= \frac{1}{\sqrt{2\pi}} \int dk'\, \widetilde{g}(k') \left[\frac{1}{\sqrt{2\pi}} \int dx\, f(x)\, e^{-i(k-k')x} \right] \\
&= \frac{1}{\sqrt{2\pi}} \int dk'\, \widetilde{g}(k')\, \widetilde{f}(k-k') \\
&= \frac{1}{\sqrt{2\pi}} (\widetilde{g} \star \widetilde{f})(k).
\end{aligned}
$$

□

There is an inverse theorem as well.

Theorem 4.3 ('Inverse' Convolution Theorem for Fourier Transforms)**.**

$$
\boxed{\widetilde{f \star g} = \sqrt{2\pi}\, \widetilde{f}\widetilde{g}}. \tag{4.4}
$$

We say that the Fourier transform of the convolution of two functions is the product of the Fourier transforms (again, up to irritating factors of $\sqrt{2\pi}$).

Proof. Take the inverse transform of both sides of the convolution theorem applied to $\widetilde{f}$ and $\widetilde{g}$. □

Here is a silly example of the use of the convolution theorem to evaluate an integral (because it is easy enough to do the integral directly).

Example 6. We wish to evaluate the following integral.

$$
h(x) = \int_{-\infty}^{\infty} e^{-(x-x')^2/2} e^{-x'^2/2}\, dx'.
$$

Define

$$f(x) := e^{-x^2/2}.$$

Then

$$h(x) = (f \star f)(x).$$

By the inverse convolution theorem,

$$\widetilde{h}(k) = \widetilde{f \star f} = \sqrt{2\pi}[\widetilde{f}(k)]^2.$$

From Example 4, we have

$$\widetilde{f}(k) = e^{-k^2/2} \quad \Rightarrow \quad [\widetilde{f}(k)]^2 = e^{-k^2},$$

so

$$\widetilde{h}(k) = \sqrt{2\pi}e^{-k^2},$$

and

$$h(x) = \frac{1}{\sqrt{2\pi}} \int \widetilde{h}(k) e^{ikx}\, dk = \int e^{-k^2} e^{ikx}\, dk = \sqrt{\pi} e^{-x^2/4}.$$

5 Hermitian Inner Product

Recall that in Section 1.2 we introduced the notion of an inner product space as a vector space over some field equipped with an inner product taking values in the field. At the time, we were only concerned with real numbers, but now we have complex numbers floating around, so we want to define a new inner product appropriate to the complex field.

Definition. *An* **inner product** *(also called a* **sesquilinear** *or* **Hermitian inner product***) on a vector space V over $\mathbb{C}$ is a map $(\cdot,\cdot) \to \mathbb{C}$ taking a pair of vectors to a complex number, satisfying the following three properties:*

$$
\begin{aligned}
&(1) \quad (u, av + bw) = a(u,v) + b(u,w) && \textit{(linearity on } 2^{nd} \textit{ entry)}\\
&(2) \quad (u,v) = (v,u)^* && \textit{(Hermiticity)}\\
&(3) \quad (u,u) \geq 0, = 0 \Leftrightarrow u = 0 && \textit{(nondegeneracy).}
\end{aligned}
$$

(Note that Property 3 makes sense, because by virtue of Property 2, (u,u) is always a real number.) Given two complex valued functions f and g, we will define

$$(f,g) = \int_{-\infty}^{\infty} f^*(x) g(x)\, dx. \tag{5.1}$$

You can check that, by this definition, (f,g) satisfies all the axioms of a Hermitian inner product.

5.1 Invariance of Inner Product: Parseval's Theorem

The naturality of the inner product (5.1) is made clear by the following beautiful theorem.

Theorem 5.1.

$$\boxed{(f,g) = (\widetilde{f}, \widetilde{g})}. \tag{5.2}$$

Before proving it, let's pause to think about what it means. Recall that the ordinary Euclidean inner (dot) product was invariant under rotations. If u and v are two vectors and R is a rotation matrix, then we observed that

$$(u,v) = (u', v'),$$

where $u' = Ru$ and $v' = Rv$. That is, rotations preserve the Euclidean inner product. Rotations are a special kind of change of basis, so we could say in this case that the inner product is preserved under a change of basis. We can view the Fourier transform (2.3) as a kind of *infinite dimensional change of basis*, in which case Theorem 5.1 says that this change of basis preserves the inner product. In other words, under Fourier transformation, the lengths and angles of infinite dimensional vectors are preserved. In this sense, the Fourier transformation doesn't really change the function, just the basis in which it is expressed. In any case, the proof of Theorem 5.1 is a simple application of the delta function.

Proof.

$$
\begin{aligned}
(f, g) &= \int f^*(x) g(x)\, dx \\
&= \int dx \left[\frac{1}{\sqrt{2\pi}} \int \widetilde{f}(k)\, e^{ikx}\, dk\right]^* \times \left[\frac{1}{\sqrt{2\pi}} \int \widetilde{g}(k')\, e^{ik'x}\, dk'\right] \\
&= \int dk\, dk'\, \widetilde{f}^*(k)\widetilde{g}(k') \times \frac{1}{2\pi} \int dx\, e^{i(k'-k)x} \\
&= \int dk\, dk'\, \widetilde{f}^*(k)\widetilde{g}(k')\delta(k' - k) \\
&= \int dk\, \widetilde{f}^*(k)\widetilde{g}(k) \\
&= (\widetilde{f}, \widetilde{g}).
\end{aligned}
$$

□

Setting $f = g$ in Theorem 5.1 yields a continuous analogue of Parseval's identity.

Corollary 5.2 (Parseval's Theorem)**.**

$$\boxed{(f, f) = (\widetilde{f}, \widetilde{f})}. \tag{5.3}$$

Equivalently,

$$\boxed{\int |f(x)|^2 \, dx = \int |\widetilde{f}(k)|^2 \, dk}. \tag{5.4}$$

5.2 Wiener-Khintchine Theorem

In this section we briefly describe an important application of Parseval's theorem to signal analysis. Let $E(t)$ represent (the electric field part of) an electromagnetic wave. Let $I(t)$ be the **instantaneous intensity (power per unit area)** carried by the wave. Then

$$I(t) = k|E(t)|^2 \tag{5.5}$$

for some constant k. Suppose we observe the wave for a finite time T (or suppose it only exists for that time). Define

$$V(t) = \begin{cases} E(t), & \text{for } |t| \leq T, \\ 0, & \text{otherwise,} \end{cases} \tag{5.6}$$

and let $\widetilde{V}(\omega)$ be its Fourier transform (as in (2.5)). The **average intensity** I over the period T can be written

$$I = \langle I(t) \rangle = \frac{k}{T} \int_{-T/2}^{T/2} |E(t)|^2 \, dt = \frac{k}{T} \int_{-\infty}^{\infty} |V(t)|^2 \, dt. \tag{5.7}$$

By Parseval's theorem

$$I = \frac{k}{T}\int_{-\infty}^{\infty} |\widetilde{V}(\omega)|^2 \, d\omega. \tag{5.8}$$

This allows us to interpret

$$P(\omega) := \frac{k}{T}|\widetilde{V}(\omega)|^2 \tag{5.9}$$

as the **power spectrum** of the signal, namely the **average power per unit frequency interval**. Basically, $P(\omega)$ tells us the amount of power that is carried in each frequency band.

There is a very interesting relation between the power spectrum of an electromagnetic wave and the amplitude of the signal at different times. More precisely, we define the following statistical averages.

Definition. *The* **cross correlation function** *of two functions* $f(t)$ *and* $g(t)$ *defined over some time interval* $[-T/2, T/2]$ *is*

$$\langle f^*(t)g(t+\tau)\rangle = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t)g(t+\tau)\, dt. \tag{5.10}$$

The cross correlation function of two functions provides information as to how the two functions correlate at times differing by a fixed amount τ.

Definition. *The* **autocorrelation function** *of* $f(t)$ *is*

$$\langle f^*(t)f(t+\tau)\rangle. \tag{5.11}$$

The autocorrelation function gives information as to how a function correlates with itself at times differing by a fixed amount τ. It is related to the degree of coherence of the signal. The **Wiener-Khintchine Theorem** basically says that the power spectrum is just the Fourier transform of the autocorrelation function of a signal. It is used in both directions. If one knows the power spectrum, one can deduce information about the coherence of the signal, and *vice versa.*

Theorem 5.3 (Wiener-Khintchine)**.** *Assume that $V(t)$ vanishes for $|t| > T$. Then*

$$\int_{-\infty}^{\infty} \langle V^*(t)V(t+\tau)\rangle e^{-i\omega\tau}\, d\tau = \frac{2\pi}{k} P(\omega). \tag{5.12}$$

Proof. We have

$$\begin{aligned}
&\langle V^*(t)V(t+\tau)\rangle \\
&\quad = \frac{1}{T}\int_{-T/2}^{T/2} V^*(t)V(t+\tau)\, dt \\
&\quad = \frac{1}{T}\int_{-\infty}^{\infty} V^*(t)V(t+\tau)\, dt \\
&\quad = \frac{1}{T}\int_{-\infty}^{\infty} dt\, V(t+\tau)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widetilde{V}(\omega)e^{i\omega t}\, d\omega\right]^* \\
&\quad = \int_{-\infty}^{\infty} d\omega\, \frac{1}{T}\widetilde{V}^*(\omega)e^{i\omega\tau}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} V(t+\tau)e^{-i\omega(t+\tau)}\, dt\right] \\
&\quad = \int_{-\infty}^{\infty} d\omega\, \frac{1}{T}\widetilde{V}^*(\omega)\widetilde{V}(\omega)e^{i\omega\tau} \\
&\quad = \int_{-\infty}^{\infty} d\omega\, \frac{1}{k}P(\omega)e^{i\omega\tau}
\end{aligned}$$

Now take the Fourier transform of both sides (and watch out for $\sqrt{2\pi}$'s). □

6 Laplace Transforms

Having discussed Fourier transforms extensively, we now turn to a different kind of integral transform that is also of great practical importance. Laplace transforms have many uses, but one of the most important is to electrical engineering and control theory, where they are used to help solve linear differential equations. They are most often applied to problems in which one has a time dependent signal defined for $0 \leq t \leq \infty$. For any complex number s, define

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{6.1}$$

This is the **Laplace transform** of $f(t)$. We also write

$$F(s) = \mathcal{L}\{f(t)\} \tag{6.2}$$

to emphasize that the Laplace transform is a kind of linear operator on functions.

6.1 Examples of Laplace Transforms

Example 7.

$$\begin{aligned} F(s) = \mathcal{L}\{t\} &= \int_0^\infty t e^{-st}\, dt = -\frac{d}{ds} \int_0^\infty e^{-st}\, dt \\ &= \frac{d}{ds} \left(\frac{1}{s} e^{-st} \right) \Bigg|_{t=0}^{\infty} = \frac{1}{s^2}. \end{aligned} \tag{6.3}$$

Observe that, if $s = \sigma + i\omega$, then

$$F(s) = \int_0^\infty t e^{-\sigma t} e^{-i\omega t}\, dt = \int_0^\infty t e^{-\sigma t} \cos \omega t\, dt - i \int_0^\infty t e^{-\sigma t} \sin \omega t\, dt$$

These integrals converge if $\sigma > 0$ and diverge if $\sigma \leq 0$. So the above equality is valid only for Re $s > 0$.

Example 8.

$$F(s) = \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-(s-a)t}\, dt = \frac{1}{s-a} \qquad (\text{Re } s > a). \tag{6.4}$$

Remark. These examples show that, in general, the Laplace transform of a function $f(t)$ exists only if Re s is greater than some real number, determined by the function $f(t)$. These limits of validity are usually included in tables of Laplace transforms.

Example 9.

$$\begin{aligned} F(s) = \mathcal{L}\{\sin at\} &= \frac{1}{2i}\mathcal{L}\{e^{iat} - e^{-iat}\} \\ &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{a}{s^2+a^2} \qquad (\text{Re } s > |\text{Im } a|) \qquad (6.5) \\ F(s) = \mathcal{L}\{\cos at\} &= \frac{1}{2}\mathcal{L}\{e^{iat} + e^{-iat}\} \\ &= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right) = \frac{s}{s^2+a^2} \qquad (\text{Re } s > |\text{Im } a|) \qquad (6.6) \end{aligned}$$

Example 10. [Laplace transform of a delta function]

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}\, dt = e^{-as} \qquad (a > 0). \tag{6.7}$$

Remark. The Laplace transform of $\delta(t)$ cannot be defined directly, because the limits of integration include zero, where the delta function is undefined. Nevertheless, by taking the (illegitimate) limit of (6.7) as $a \to 0$, we may define

$$\mathcal{L}\{\delta(t)\} = 1. \tag{6.8}$$

Example 11. The **gamma function** $\Gamma(z)$ is defined by

$$\boxed{\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt}. \tag{6.9}$$

This integral exists for all complex z with Re $z > 0$. It follows that

$$\mathcal{L}\{t^z\} = \int_0^\infty t^z e^{-st}\, dt = \frac{\Gamma(z+1)}{s^{z+1}} \qquad (\text{Re } z > -1, \text{Re } s > 0). \tag{6.10}$$

The gamma function shows up a lot in math and physics, so it is worth noting a few of its properties. Integration by parts gives

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt = -\left. t^z e^{-t} \right|_{t=0}^\infty + z \int_0^\infty t^{z-1} e^{-t}\, dt = z\Gamma(z),$$

whence we obtain the fundamental functional equation

$$\boxed{\Gamma(z+1) = z\Gamma(z)}, \tag{6.11}$$

for all complex z where the integral is defined. Also

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -\left. e^{-t} \right|_0^\infty = 1, \tag{6.12}$$

so if $z = n$, a natural number, then

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!. \tag{6.13}$$

Hence, the gamma function generalizes the usual factorial function. Note that

$$\Gamma(1/2) = \sqrt{\pi}, \tag{6.14}$$

which is proved by changing variables from z to x^2 in the defining integral (6.9) and using (2.10):

$$\begin{aligned}\Gamma(1/2) &= \int_0^\infty z^{-1/2} e^{-z}\, dz = \int_0^\infty x^{-1} e^{-x^2} (2x)\, dx = 2\int_0^\infty e^{-x^2}\, dx \\ &= \int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.\end{aligned}$$

6.2 Basic Properties of Laplace Transforms

The most obvious property of Laplace transforms is that any two functions agree for $t > 0$ have the same Laplace transform. This is sometimes used to advantage in signal processing. Recall the definition of the **step function**:

$$\theta(x) := \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0. \end{cases} \tag{6.15}$$

It follows that, for any function $f(t)$,

$$\boxed{\mathcal{L}\{f(t)\} = \mathcal{L}\{\theta(t) f(t)\}} \qquad \text{(chopping property).} \tag{6.16}$$

We also have, for $a > 0$,

$$\begin{aligned}\mathcal{L}\{f(t-a)\theta(t-a)\} &= \int_0^\infty f(t-a)\theta(t-a)e^{-st}\,dt \\ &= \int_a^\infty f(t-a)e^{-st}\,dt \\ &= \int_0^\infty f(x)e^{-s(x+a)}\,dx,\end{aligned}$$

or

$$\boxed{\mathcal{L}\{f(t-a)\theta(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} \quad (a > 0)} \qquad \text{(shifting property).} \tag{6.17}$$

Similarly, we have

$$\boxed{\mathcal{L}\{e^{-at}f(t)\} = F(s+a) \quad (\text{Re } s > -a)} \qquad \text{(attenuation property).} \tag{6.18}$$

Example 12. Combining (6.3) and the attenuation theorem gives

$$F(s) = \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2} \qquad (\text{Re } s > a). \tag{6.19}$$

One of the most important properties of the Laplace transform is the relation between $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{\dot{f}(t)\}$, where $\dot{f}(t) = df/dt$. Integration by parts shows that

$$\boxed{\mathcal{L}\{\dot{f}(t)\} = s\mathcal{L}\{f(t)\} - f(0)} \qquad \text{(derivative property).} \tag{6.20}$$

Higher derivatives are obtained by simple substitution. For instance,

$$\mathcal{L}\{\ddot{f}(t)\} = s\mathcal{L}\{\dot{f}(t)\} - \dot{f}(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - \dot{f}(0). \tag{6.21}$$

6.3 Inverting Laplace Transforms

An example will illustrate the way in which Laplace transforms are typically used. Suppose we want to find a solution to the differential equation

$$\dot{f} + f = e^{-t}.$$

Multiply the equation by e^{-st} and integrate from 0 to ∞ (Laplace transform the equation) to get

$$\mathcal{L}\{\dot{f}\} + \mathcal{L}\{f\} = \mathcal{L}\{e^{-t}\}.$$

Define $\mathcal{L}\{f(t)\} = F(s)$ and use the derivative property together with the inverse of (6.4) to get

$$sF(s) - f(0) + F(s) = \frac{1}{s+1}.$$

Essentially, we have reduced a differential equation to an algebraic one that is much easier to solve. Indeed, we have

$$F(s) = \frac{1}{s+1}f(0) + \frac{1}{(s+1)^2}. \tag{6.22}$$

To solve the original equation we must *invert* $F(s)$ to recover the original function $f(t)$. This is usually done by consulting a table of Laplace transforms. In our case, the inverse of (6.22) can be computed from the inverses of (6.4) and (6.19):

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = f(0)e^{-t} + te^{-t}. \tag{6.23}$$

In general, to invert a Laplace transform we must employ something called a **Mellin inversion integral**. But in simple cases we can avoid this integral by means of some simple tricks. Two such tricks are particularly important.

6.3.1 Partial Fractions

Again, a simple example illustrates the point.

Example 13. Given $F(s) = (s^2 - 5s + 6)^{-1}$, find $f(t)$. We have

$$F(s) = \frac{1}{(s-2)(s-3)} = \frac{1}{s-3} - \frac{1}{s-2},$$

so by the inverse of (6.4),

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{3t} - e^{2t}.$$

Many problems give rise to a Laplace transform that is the ratio of two polynomials

$$F(s) = \frac{P(s)}{Q(s)}. \tag{6.24}$$

Without loss of generality we may assume that the degree of $P(s)$ is less than the degree of $Q(s)$, because if not, we can use polynomial division to reduce the ratio to a polynomial plus a remainder term of this form. [7] In

[7] In any case, polynomials do not possess well-behaved inverse Laplace transforms.

that case, we may use the technique of partial fractions to rewrite $F(s)$ in a form enabling one to essentially read off the inverse transform from a table.

There are several ways to do partial fraction expansions, but some are more cumbersome than others. In general, a ratio of polynomials of the form

$$\frac{p(x)}{q(x)}, \tag{6.25}$$

where

$$q(x) = (x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_k)^{n_k}, \tag{6.26}$$

and all the a_i's are distinct, and where the degree of $p(x)$ is less than $\sum n_i$, admits a partial fraction expansion of the form [8]

$$\begin{aligned} &\frac{A_{1,n_1}}{(x-a_1)^{n_1}} + \frac{A_{1,n_1-1}}{(x-a_1)^{n_1-1}} + \cdots + \frac{A_{1,1}}{(x-a_1)} + \cdots \\ &\quad + \frac{A_{2,n_2}}{(x-a_2)^{n_2}} + \frac{A_{2,n_2-1}}{(x-a_2)^{n_2-1}} + \cdots + \frac{A_{2,1}}{(x-a_2)} + \cdots \\ &\quad + \frac{A_{k,n_k}}{(x-a_k)^{n_k}} + \frac{A_{k,n_k-1}}{(x-a_k)^{n_k-1}} + \cdots + \frac{A_{k,1}}{(x-a_k)}. \end{aligned} \tag{6.27}$$

There are now several different methods to find the $A_{i,j}$, which we present as a series of examples.

Example 14. [Equating Like Powers of s] Combine the terms on the right hand side into a polynomial, and equate coefficients. (This was the method used in (13).) Write

$$\frac{1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}.$$

[8] Observe that this *assumes* that that one knows how to factor $q(x)$, a problem for which there is no general algorithm! One method is to guess a root, then find the remainder term by polynomial division and repeat.

Cross multiply by $(s-2)(s-3)$ to get

$$1 = A(s-3) + B(s-2) = (A+B)s - (3A+2B).$$

Equate like powers of s on both sides to get

$$A + B = 0$$
$$3A + 2B = -1.$$

Solve (either by substitution or linear algebra) to get $A = -1$ and $B = 1$, as before.

Example 15. [Plugging in Values of s] This method is a minor variation on the previous one. Consider the expansion

$$\frac{2s+6}{(s-3)^2(s+1)} = \frac{A}{(s-3)^2} + \frac{B}{(s-3)} + \frac{C}{s+1}.$$

Cross multiply by $q(x)$ as before to get

$$2s + 6 = A(s+1) + B(s-3)(s+1) + C(s-3)^2.$$

Now plug in judicious values of s. For example, if we plug in $s = 3$, two terms on the right disappear, and we immediately get $12 = 4A$ or $A = 3$. Then try $s = -1$. This gives $4 = 16C$ or $C = 1/4$. It remains only to find B, which we can do by plugging in the values of A and C we already found and equating coefficients of s^2 on both sides. This gives $B = -1/4$. Hence

$$\frac{2s+6}{(s-3)^2(s+1)} = \frac{3}{(s-3)^2} - \frac{1}{4(s-3)} + \frac{1}{4(s+1)}.$$

Example 16. [Taking Derivatives] Suppose

$$q(x) = (x-a)^n r(x), \tag{6.28}$$

where $r(x)$ is a polynomial with $r(a) \neq 0$. Then we have

$$\frac{p(x)}{q(x)} = \frac{\alpha_n}{(x-a)^n} + \frac{\alpha_{n-1}}{(x-a)^{n-1}} + \cdots + \frac{\alpha_1}{x-a} + f(x), \tag{6.29}$$

where $f(x)$ means all the other terms in (6.27). (Notice that, whatever those terms are, they do not blow up at $x = a$, because they involve fractions with denominators that are finite at $x = a$.) To find α_k, we can do the following. First, multiply (6.29) through by $(x-a)^n$. This gives

$$\frac{p(x)}{r(x)} = \alpha_n + \alpha_{n-1}(x-a) + \alpha_1(x-a)^{n-1} + h(x), \tag{6.30}$$

where $h(x) = (x-a)^n f(x)$. Now take k derivatives of both sides, where $0 \leq k \leq n-1$, and evaluate at $x = a$. As $f(a) < \infty$, all those derivatives of $h(x)$ vanish at $x = a$, and what remains is $k!\alpha_{n-k}$. Therefore,

$$\boxed{\alpha_{n-k} = \frac{1}{k!}\frac{d^k}{dx^k}\frac{p(x)}{r(x)}\bigg|_{x=a} \qquad (0 \leq k \leq n-1)}. \tag{6.31}$$

To find the other terms $f(x)$ in the expansion (6.29), just perform the same trick at all the other roots of $q(x)$.

For example, let's use this method to compute

$$\mathcal{L}^{-1}\left\{\frac{1+s+s^2}{s^2(s-1)^2}\right\}.$$

We have

$$\frac{1+s+s^2}{s^2(s-1)^2} = \frac{\alpha_2}{s^2} + \frac{\alpha_1}{s} + \frac{\beta_2}{(s-1)^2} + \frac{\beta_1}{s-1},$$

so applying (6.31) twice, first with $r(s) = (s-1)^2$ and then with $r(s) = s^2$, gives

$$\begin{aligned}
\alpha_2 &= \frac{1}{0!}\,\frac{1+s+s^2}{(s-1)^2}\bigg|_{s=0} = 1 \\
\alpha_1 &= \frac{1}{1!}\frac{d}{ds}\,\frac{1+s+s^2}{(s-1)^2}\bigg|_{s=0} = 3 \\
\beta_2 &= \frac{1}{0!}\,\frac{1+s+s^2}{s^2}\bigg|_{s=1} = 3 \\
\beta_1 &= \frac{1}{1!}\frac{d}{ds}\,\frac{1+s+s^2}{s^2}\bigg|_{s=1} = -3.
\end{aligned}$$

Thus,

$$\frac{1+s+s^2}{s^2(s-1)^2} = \frac{1}{s^2} + \frac{3}{s} + \frac{3}{(s-1)^2} - \frac{3}{s-1},$$

whereupon we obtain

$$\mathcal{L}^{-1}\left\{\frac{1+s+s^2}{s^2(s-1)^2}\right\} = t + 3 + 3te^t - 3e^t.$$

Example 17. [Recursion] A variant on the previous method that avoids derivatives was provided by van der Waerden in his classic textbook on algebra. It is recursive in nature and provably [9] fastest, which makes it particularly well suited to automation, although for hand calculation the derivative method is probably easiest. As in the derivative method, we want to expand

[9] H.J. Straight and R. Dowds, "An Alternate Method for Finding the Partial Fraction Decomposition of a Rational Function", *Am. Math. Month.* **91:6** (1984) 365-367.

$p(x)/q(x)$ where $p(x)$ and $q(x)$ are relatively prime, deg $p(x) <$ deg $q(x)$, and $q(x) = (x-a)^n r(x)$, where $r(a) \neq 0$ and $n \geq 1$. For any constant c we can write

$$\frac{p(x)}{q(x)} = \frac{c}{(x-a)^n} + \frac{p(x) - cr(x)}{(x-a)^n r(x)}. \tag{6.32}$$

Now we choose $c = p(a)/r(a)$. Then by the division algorithm for polynomials

$$p(x) - cr(x) = (x-a)p_1(x), \tag{6.33}$$

for some polynomial $p_1(x)$ with deg $p_1(x) <$ deg $p(x)$, because a is a root of the left hand side. Thus we can write

$$\frac{p(x)}{q(x)} = \frac{c}{(x-a)^n} + \frac{p_1(x)}{(x-a)^{n-1} r(x)}. \tag{6.34}$$

Now apply the same technique to the second term, and so on. Eventually we obtain the entire partial fraction expansion.

Consider again

$$F(s) := \frac{1+s+s^2}{s^2(s-1)^2}.$$

We begin by taking $p(s) = 1 + s + s^2$, $a = 0$ and $r(s) = (s-1)^2$. Applying the above method gives $c = p(a)/r(a) = 1$ and

$$p_1(s) = \frac{p(s) - r(s)}{s} = \frac{(1+s+s^2) - (s^2 - 2s + 1)}{s} = 3$$

so

$$F(s) = \frac{1}{s^2} + \frac{3}{s(s-1)^2}$$

For the next step we again take $a = 0$ and $r(s) = (s-1)^2$. Then $c =$

$p_1(a)/r(a) = 3$ and

$$p_2(s) = \frac{p_1(s) - 3r(s)}{s} = \frac{3 - 3(s^2 - 2s + 1)}{s} = -3s + 6$$

so

$$F(s) = \frac{1}{s^2} + \frac{3}{s} + \frac{-3s + 6}{(s-1)^2}$$

Next take $a = 1$ and $r(s) = 1$. Then $c = p_2(1)/r(1) = 3$ and

$$p_3(s) = \frac{p_2(s) - 3r(s)}{s - 1} = \frac{-3s + 3}{s - 1} = -3$$

so

$$\boxed{F(s) = \frac{1}{s^2} + \frac{3}{s} + \frac{3}{(s-1)^2} + \frac{-3}{s-1}}$$

as before. Pretty slick!

6.3.2 The Laplace Convolution Theorem

Just as for Fourier transforms, there is a convolution theorem for Laplace transforms that sometimes comes in handy. Unfortunately, we need to introduce a slightly different definition of convolution, appropriate for Laplace transforms. To distinguish it from the one for Fourier transforms, we denote it using the symbol '$*$'. You must be very careful to keep track of which convolution function you are using in any particular application!

Definition. *The convolution of two functions $f(t)$ and $g(t)$ is*

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau. \tag{6.35}$$

As before, although it is not obvious, convolution is symmetric: $f * g = g * f$.

Theorem 6.1 (Convolution Theorem for Laplace Tranforms)**.** *If $f(t)$ and $g(t)$ are sufficiently well behaved, with $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$, then*

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s). \tag{6.36}$$

Hence,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \tag{6.37}$$

Proof. Omitted. □

Example 18. Using the inverse of (6.4) and the definition (6.35), we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} &= e^{at} * e^{bt} = \int_0^t e^{a\tau}e^{b(t-\tau)}\,d\tau \\ &= \frac{e^{at}-e^{bt}}{a-b} \qquad (a \neq b).\end{aligned} \tag{6.38}$$

Example 19. From (6.10) and (6.14) we have

$$\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{1/2}} = \sqrt{\frac{\pi}{s}}. \tag{6.39}$$

Hence

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s-1)}\right\} = \frac{1}{\sqrt{\pi t}} * e^t = \int_0^t \frac{1}{\sqrt{\pi\tau}}e^{t-\tau}\,d\tau = \frac{e^t}{\sqrt{\pi}}\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau$$

Change variables from τ to $x = \sqrt{\tau}$ to get

$$\int_0^t \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau = 2\int_0^{\sqrt{\tau}} e^{-x^2}\,dx.$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}(s+1)}\right\} = e^t \operatorname{erf}(\sqrt{t}), \tag{6.40}$$

where

$$\boxed{\operatorname{erf}(u) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-x^2}\,dx} \tag{6.41}$$

is the **error function** of probability theory.

Example 20. Here is a slick application of the convolution theorem. The **beta function**, defined for Re $x > 1$, Re $y > 1$ is

$$\boxed{B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt}. \tag{6.42}$$

Note that the integral is symmetric in x and y (change variables from t to $1-t$). Let $f(t) = t^{a-1}$ and $g(t) = t^{b-1}$ with $a > 0$ and $b > 0$. Then, with the substitution $\tau = ut$ we can write

$$(f*g)(t) = \int_0^t \tau^{a-1}(t-\tau)^{b-1}\,d\tau = t^{a+b-1}\int_0^1 u^{a-1}(1-u)^{b-1}\,du = t^{a+b-1}B(a,b).$$

By the convolution theorem and (6.10),

$$\mathcal{L}\{t^{a+b-1}B(a,b)\} = \mathcal{L}\{t^{a-1} * t^{b-1}\} = \mathcal{L}\{t^{a-1}\}\mathcal{L}\{t^{b-1}\} = \frac{\Gamma(a)\Gamma(b)}{s^{a+b}}.$$

Using the inverse of (6.10) we obtain

$$t^{a+b-1}B(a,b) = \Gamma(a)\Gamma(b)\mathcal{L}^{-1}\left\{\frac{1}{s^{a+b}}\right\} = \Gamma(a)\Gamma(b)\frac{t^{a+b-1}}{\Gamma(a+b)}.$$

Hence,

$$\boxed{B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}, \tag{6.43}$$

a neat formula due to Euler.