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A Bound on the Number of Minimum Covers of a Hypergraph

Paul Renteln
Department of Physics
California State University
San Bernardino, CA 92407
prenteln@csusb.edu
and
Department of Mathematics
California Institute of Technology
Pasadena, CA 91125

ABSTRACT

We generalize a result of Gyárfás on the number of minimum covers of a hypergraph.

KEYWORDS: hypergraph, covering, independent set.

Let $|X| = n$, and let $\mathcal{F} := \{A_1, A_2, \ldots, A_m\}$ be a simple hypergraph on X. A *(vertex) cover* (or *blocking set* or *transversal*) of $\mathcal{F}$ is a set $B \subseteq X$ satisfying $B \cap A \neq \emptyset$ for all $A \in \mathcal{F}$. A cover is *minimal* if it contains no other cover, and it is *minimum* if it has the least cardinality among all minimal covers. This minimum cardinality τ is called the *covering number* of $\mathcal{F}$.

An *independent set* of a hypergraph $\mathcal{F}$ is a set J that contains no edge of $\mathcal{F}$. J is *maximal* if it is contained in no other independent set, and it is *maximum* if it has the greatest cardinality among all maximal independent sets. This maximum cardinality α is called the *independence number* of $\mathcal{F}$. Observe that $\alpha + \tau = n$, because the complement of a maximal independent set is a minimal cover (any edge not meeting the complement of a set Y must be contained in Y).

The *rank* of a hypergraph $\mathcal{F}$ is the maximum cardinality of an edge. In 1977 Gyárfás [2] proved (see also [1], [3]):

THEOREM 1. *Let $\mathcal{F}$ be a hypergraph of rank r and covering number τ. Then the number of minimum covers of $\mathcal{F}$ is at most r^τ.*

The purpose of this note is to offer a small improvement on Theorem 1 when the hypergraph is not uniform. (A hypergraph is uniform if every edge has the same cardinality.) The method is basically dual to that used by Gyárfás.

THEOREM 2. *Let $\mathcal{F} := \{A_1, A_2, \ldots, A_m\}$ be a hypergraph with covering number τ and edges listed in order of nonincreasing cardinality. Then the number of minimum covers of $\mathcal{F}$ is at most $\prod_{i=1}^{\tau} |A_i|$.*

Proof. We show that the number of maximum independent sets of $\mathcal{F}$ is at most $\prod_{i=1}^{\tau} |A_i|$. Let $I \subseteq X$ be a set of size $\alpha + j$. The collection of all edges of $\mathcal{F}$ that are contained in I is denoted by $\mathcal{F}' := \{A'_1, A'_2, \ldots, A'_s\}$. Again, the sets are listed in order of nonincreasing cardinality. We show by induction on j that I contains at most $\prod_{i=1}^{j} |A'_i|$ maximum independent sets of $\mathcal{F}$, where this quantity is defined to be 1 when $j = 0$. Then the base case $j = 0$ is trivial, and the case $j = \tau$ gives the theorem. Note that the product is always defined, because we must have $s \geq j$, otherwise we can remove a point from each of the sets in $\mathcal{F}'$ and obtain an independent set of size greater than α.

We count the number of ways of removing points from I to reach a maximum independent set. Let $x \in A'_1$ and let $\mathcal{F}'' := \{A''_2, A''_3, \ldots, A''_r\}$ be the sets of $\mathcal{F}'$ that do not contain x, listed in order of nonincreasing cardinality. Observe that $|A''_i| \leq |A'_i|$ for all $2 \leq i \leq r$. By hypothesis, there are at most $\prod_{i=2}^{j} |A''_i|$ ways to remove points of $I - x$ and reach a maximum independent set of $\mathcal{F}$. (Note that, due to the index shift in $\mathcal{F}''$, the indices on the above product are correct.) Hence there are at most $|A'_1| \prod_{i=2}^{j} |A''_i| \leq \prod_{i=1}^{j} |A'_i|$ maximum independent sets of $\mathcal{F}$ in I. ∎

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