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The Hilbert Series of the Face Ring of a Flag Complex

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ABSTRACT

It is shown that the Hilbert series of the face ring of a clique complex (equivalently, flag complex) of a graph G is, up to a factor, just a specialization of $S_{\overline{G}}(x, y)$, the subgraph polynomial of the complement of G. We also find a simple relationship between the size of a minimum vertex cover of a graph G and its subgraph polynomial. This yields a formula for the h-vector of the flag complex in terms of those two invariants of $\overline{G}$. Some computational issues are addressed and a recursive formula for the Hilbert series is given based on an algorithm of Bayer and Stillman.

KEYWORDS: face ring, flag complex, clique complex, Hilbert series, subgraph polynomial, vertex cover

1. Introduction

Associated to every graph G are several natural simplicial complexes. Of these, one of the most basic is the clique complex $\Delta(G)$, whose faces are the cliques of G. Many interesting mathematical problems can be cast in terms of the number of faces of a clique complex of a graph. In this paper we enumerate these face numbers in terms of the subgraph polynomial $S_{\overline{G}}(x,y)$ of $\overline{G}$, the complement of G. As a corollary, we deduce an expression for the minimum vertex cover of a graph in terms of its subgraph polynomial.

Our results follow from a computation of the Hilbert series of the face ring of the clique complex. To carry out this computation we use a recent result of Gasharov, Peeva, and Welker [1], which expresses the Betti numbers of the face ring in terms of the homology of a certain lattice of monomials related to the complex (the so-called *lcm-lattice*). To compute the Hilbert series only the Euler characteristics of lower intervals of the lcm-lattice are needed, and these turn out to be directly related to a specialization of the subgraph polynomial.

We begin by reviewing the basic ideas underlying our construction.

Flag Complexes

Let Δ be a finite simplicial complex on the vertex set $V = \{1, \ldots, n\}$. That is, Δ is a collection of subsets of V such that $F' \in \Delta$ and $F \subseteq F'$ implies $F \in \Delta$, and $\{i\} \in \Delta$ for all i. Elements of Δ are called *faces*, and the *dimension* of a face is one less than its cardinality. An r-dimensional face is called an *r-face* for short. The dimension of Δ is the dimension of a maximal face. Let $f_i(\Delta)$ be the number of i-faces of Δ. [1] If $\dim \Delta = d-1$ the d-tuple $(f_0, \ldots, f_{d-1})$ is called the *f-vector* of Δ.

A *flag complex* is a simplicial complex with the property that every minimal nonface has precisely two elements. [2] Flag complexes are intimately related to graphs. Let V be a subset of the vertices of a graph G. V is a *clique* if every pair of elements of V is joined by an edge of G, and V is *independent* if no pair of elements of V is joined by an edge of G. The collection $\Delta(G)$ of all the cliques of a graph G forms a simplicial complex, called the *clique complex* of G, by letting the r faces of $\Delta(G)$ be the cliques of size $r+1$. It is easy to see that every flag complex is the clique complex of some graph. Dually, we could

[1] Note that, as $\dim \emptyset = -1$, $f_{-1} = 1$ unless Δ is itself empty, in which case $f_i = 0$ for all i.
[2] The terminology is evidently due to Tits ([2], p.2). For a survey of some results related to flag complexes, see ([3], pp. 100ff).

also consider $\overline{\Delta}(G)$, the *independent set complex* (or *stable set complex*) of G, which is the simplicial complex formed by by letting the r faces of $\overline{\Delta}(G)$ be the independent sets of G of size $r+1$. Clearly, $\Delta(\overline{G}) = \overline{\Delta}(G)$, where $\overline{G}$ is the complement of G. Hence every flag complex is also the independent set complex of some graph.

1.1 EXAMPLE. For any poset P, its *order complex* $\Delta(P)$ (the simplicial complex whose faces are the chains (totally ordered subsets) of P) is a flag complex. To see this, let G_P be the *comparability graph* of P, namely the graph on the vertices of P whose edges are all pairs of comparable vertices. Then $\Delta(P) = \Delta(G_P)$.

1.2 EXAMPLE. Let Δ' be any simplicial complex, and let $P(\Delta')$ be its *face poset*, namely the set of faces of Δ' partially ordered by inclusion. The order complex $\Delta(P(\Delta'))$ is just the barycentric subdivision of Δ'. Hence the barycentric subdivision of any simplicial complex is a flag complex.

Many important problems reduce to questions about the $f-$vector of flag complexes.

1.3 EXAMPLE. Let G be an f_0 vertex simple graph whose clique complex $\Delta(G)$ has $f-$vector $f = (f_0, f_1, \ldots, f_{r-1})$. Then Turán's Theorem implies, for example, that $f_1 \leq (1 - 1/r)(f_0)^2/2$. Finding other inequalities on the components of the $f-$vector is of considerable interest to graph theorists.

1.4 EXAMPLE. (For further details about this example, see ([3], pp. 100ff).) Let Δ be a non-acyclic Gorenstein flag complex of dimension $2m-1$ with $f-$vector $(f_0, \ldots, f_{2m-1})$. Then Charney and Davis conjecture [3] ([4]) that

$$(-1)^m \sum_{i=0}^{2m} f_{i-1} \left(-\frac{1}{2}\right)^i \geq 0$$

THE FACE RING

A particularly effective way to investigate the f-vector of a simplicial complex is to study the *face ring* (or *Stanley-Reisner ring*) $k[\Delta]$ of the complex Δ ([3],[5],[6]). Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. To every collection $F = \{i_1, i_2, \ldots, i_r\}$ of r distinct vertices of a simplicial complex Δ we associate a monomial $x^F \in R$, where

$$x^F := x_{i_1} x_{i_2} \ldots x_{i_r} \tag{1.1}$$

[3] This conjecture is a piecewise-linear analogue of the Hopf conjecture that the Euler characteristic of a closed Riemannian $2m-$dimensional manifold with non-positive sectional curvature alternates in sign with m.

The face ring of Δ is defined to be the quotient ring R/I_Δ, where I_Δ is the ideal generated by all monomials x^F such that F is *not* a face of Δ.

The importance of the face ring lies in the fact that one may read off certain enumerative invariants of the simplicial complex Δ from certain algebraic properties of the face ring $k[\Delta]$. For example, one can show ([3], Theorem 1.3, p. 53; [6], Theorem 5.1.4, p. 202) that $\dim k[\Delta] = 1 + \dim \Delta$, where $\dim k[\Delta]$ denotes the Krull dimension of the ring. To describe these invariants more fully, we need to recall some algebraic facts.

The Hilbert Series of a Graded Module

Because R is a polynomial ring over a field, it has additional structure. For $\alpha \in \mathbb{N}^n$ we define $R_\alpha = \{cx^\alpha | c \in k\}$, where $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and x^α is shorthand for $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Then $R_0 = k$, $R = \oplus_{\alpha \in \mathbb{N}^n} R_\alpha$ (vector space direct sum), and $R_\alpha R_\beta \subseteq R_{\alpha+\beta}$. Hence R is a finitely generated $\mathbb{N}^n$-graded k-algebra. Elements of R_α are said to be *homogeneous of multidegree* α. We say that elements of R are *graded by multidegree* or *finely graded*, and we refer to the $\mathbb{N}^n$ grading as the *fine grading*.

A $\mathbb{Z}^m$-graded R-module M admits a direct sum decomposition $M = \oplus_{\alpha \in \mathbb{Z}^n} M_\alpha$ compatible with the grading: $R_\alpha M_\beta \subseteq M_{\alpha+\beta}$. In particular, each homogeneous component M_α is a module over $R_0 = k \subset R$ and is thus a k vector space. If M is finitely generated (the only case with which we shall be concerned) the k-dimension of each M_α is finite. This allows us to define the *Hilbert function* of M, which keeps track of the dimension of each homogeneous component:

$$H(M, \alpha) := \dim_k M_\alpha \tag{1.2}$$

The generating function for the Hilbert function is the *Hilbert series* of M:

$$H_M(t) = \sum_{\alpha \in \mathbb{Z}^n} H(M, \alpha) t^\alpha \tag{1.3}$$

where $t = (t_1, \ldots, t_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and t^α is shorthand for $t_1^{\alpha_1} \ldots t_n^{\alpha_n}$.

As I_Δ inherits the grading of R, the face ring $k[\Delta]$ does as well: $(R/I_\Delta)_\alpha = R_\alpha / I_{\Delta\alpha}$. To compute the Hilbert series of $k[\Delta]$ (as a multigraded R module) we proceed as follows. The *support* of a monomial x^α (equivalently, the support of α) is $\{i | \alpha_i \neq 0\}$. The nonzero monomials x^α in $k[\Delta]$ are just the ones whose support is a face of Δ. So each homogeneous component $k[\Delta]_\alpha$ is either a zero or one dimensional vector space depending on whether or not the support of α lies in Δ. Hence

$$\begin{aligned} H_{k[\Delta]}(t) &= \sum_{\substack{\alpha\in\mathbb{N}^n \\ \text{supp } \alpha\in\Delta}} t^\alpha \\ &= \sum_{F\in\Delta} \sum_{\substack{\alpha\in\mathbb{N}^n \\ \text{supp } \alpha=F}} t^\alpha \\ &= \sum_{F\in\Delta} \prod_{i\in F} \frac{t_i}{1-t_i} \end{aligned} \tag{1.4}$$

To connect the properties of the Hilbert series of $k[\Delta]$ with the properties of Δ we pass to a coarser grading of the face ring. Define the $\mathbb{Z}$-graded (*coarse graded*) component $k[\Delta]_i$ by

$$k[\Delta]_i = \bigoplus_{\alpha\in\mathbb{Z}^n,\, |\alpha|=i} k[\Delta]_\alpha \tag{1.5}$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_n$. It follows that the Hilbert series of $k[\Delta]$ with respect to the coarse grading may be obtained from (1.4) by replacing each t_i by t. This gives us the following

1.5 THEOREM ([3], Theorem 1.4, p. 54; [6], Theorem 5.1.7, p. 204). *Let Δ be a simplicial complex with f-vector $(f_0, \ldots, f_{d-1})$. Then*

$$H_{k[\Delta]}(t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}$$

It is known ([6], Corollary 4.1.8, p. 149) that the Hilbert series of any finitely generated $\mathbb{Z}$-graded R-module $M = \oplus_i M_i$ of dimension d can be written in the form

$$H_M(t) = \frac{Q_M(t)}{(1-t)^d} \tag{1.6}$$

where $Q_M(t) = \sum h_i t^i$ is a Laurent polynomial with integer coefficients such that $\min\{i|h_i \neq 0\} = \min\{i|M_i \neq 0\}$. As the face ring $k[\Delta]$ (with its standard grading) only has homogeneous components of non-negative degree, $Q_{k[\Delta]}(t)$ is an ordinary polynomial, often called the *h-polynomial* of Δ. [4] Comparing Theorem 1.5 with Equation (1.6) we get

$$\sum h_i t^i = \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} \tag{1.7}$$

[4] This is not to be confused with the *Hilbert polynomial*, which is the polynomial function of u which agrees with $H(M,u)$ for large u when M is $\mathbb{Z}$-graded.

whence it follows that $h_i = 0$ for $i > d$. The $d+1$-tuple $h(\Delta) := (h_0, \ldots, h_d)$ is called the *h-vector* of Δ. It is clear that knowing the h-vector is equivalent to knowing the f-vector. Explicitly we have

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \quad \text{and} \quad f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i \tag{1.8}$$

Hilbert Series and Free Resolutions

We can compute the Hilbert series of a graded module from its finite free resolution: [5]

1.6 Theorem ([6], Lemma 4.1.13, p. 153). *Let M be a finite graded R-module of finite projective dimension, and let*

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{pj}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0$$

be a graded free resolution of M. Then

$$H_M(t) = S_M(t) H_R(t)$$

where $S_M(t) = \sum_{ij} (-1)^i \beta_{ij} t^j$. In particular, if $R = k[x_1, \ldots, x_n]$ is the polynomial ring over the field k, then

$$H_M(t) = \frac{S_M(t)}{(1-t)^n}$$

When the resolution is minimal the numbers β_{ij} are called the (j-graded or coarsely graded) *Betti numbers* of the module M, and can be expressed as

$$\beta_{ij}(M) = \dim_k \mathrm{Tor}^R_{i,j}(M, k) \tag{1.9}$$

One important fact which we will use in the sequel is that one may read off the codimension of the module M from the Hilbert series ([6], Corollary 4.1.14, p. 153):

$$n - d = \inf\{i : d^i S_M(t)/dt^i|_{t=1} \neq 0\} \tag{1.10}$$

[5] Recall that if $M = \oplus_i M_i$ is a graded R-module with the standard grading ($\deg x_i = 1$) the notation $M(j)$, $j \in \mathbb{Z}$ means the same graded module with the degrees shifted: $M(j) = \oplus_i [M(j)]_i$ where $[M(j)]_i = M_{i+j}$. The shifted grading ensures that the connecting homomorphisms are degree preserving.

2. The Betti Numbers of a Monomial Ideal

In Section 4 we will show that the polynomial $S_M(t)$ appearing in Theorem 1.6 has a very natural interpretation in terms of the graph G when we take $M = k[\Delta(G)]$. The key tool we shall use is a result of Gasharov, Peeva, and Welker ([1]; see also [7] for further details and applications) expressing the multigraded Betti numbers of a monomial ideal in terms of the homology of a certain lattice associated to the ideal.

Let $R = k[x_1, \dots, x_n]$ be the polynomial ring over a field k with its $\mathbb{N}^n$ grading as above and let I be an ideal generated by a set of monomials $m_1, \dots, m_s$. Denote by L_I the lattice whose elements are the least common multiples of subsets of the generating set of I ordered by divisibility. Thus the atoms of L_I are the monomials $m_1, \dots, m_s$, the minimal element of L_I is 1 (corresponding to the least common multiple of the empty set), and the maximal element is $\mathrm{lcm}(m_1, \dots, m_s)$. L_I is called the *lcm lattice* of the ideal I.

The main result which we shall need is

2.1 Theorem ([1], Theorem 2.1). *For $i \geq 1$ and $x^\alpha \in L_I$ we have*

$$\beta_{i,\alpha}(R/I) = \dim\, \widetilde{H}_{i-2}\left((0, x^\alpha)_{L_I}; k\right)$$

Here $\widetilde{H}_j(X; k)$ is the j^{th} reduced simplicial homology group of the space X over k, and $(0, x^\alpha)$ denotes the (open) lower interval between the minimal element and x^α in L_I. (The topology of a poset is the topology of its order complex.) The numbers $\beta_{i,\alpha}(R/I)$ are the finely graded Betti numbers of a minimal finite free resolution of R/I. As noted in [1], Taylor's resolution ([8], Exercise 17.11, p. 439) shows that the finely graded Betti numbers vanish if the monomial $x^\alpha \notin L_I$. Also, as the total degree of a monomial is just the sum of the entries of its multidegree (*cf.* (1.5)) we have

$$\beta_{ij} = \sum_{\substack{\alpha \in \mathbb{Z}^n \\ |\alpha| = j}} \beta_{i,\alpha} \tag{2.1}$$

3. The Subgraph Polynomial

To every graph G we may associate its *subgraph polynomial* $S_G(x, y)$ defined as follows

$$S_G(x, y) := \sum_{T \subseteq EG} x^{|T|} y^{|VT|} \tag{3.1}$$

where EG is the edge set of G, $|T|$ is the cardinality of the set T, and VT is the subset of vertices of G incident with the edges in T. Alternatively, we may write

$$S_G(x,y) = \sum_{ij} b_{ij} x^i y^j \tag{3.2}$$

where b_{ij} is the number of subgraphs of G with i edges and j vertices. [6]

Variants of this polynomial have been considered by Farrell [9] and Borzacchini [10] and probably others, although considering its naturality there are suprisingly few references to it in the literature. One possible reason for this dearth of references is that the subgraph polynomial is *not* a generalized Tutte-Grothendieck invariant (in the sense of Brylawski [11]; see below). Hence it is not simply a specialization of the Tutte polynomial of the graph. [7]

The subgraph polynomial does satisfy a simple recursion formula based on deletion and contraction of edges, which we record here. Given a graph G and an edge $e \in EG$ we let $G - e$ denote the graph G with e removed ("the deletion of e") and we let G/e denote the graph G with e removed and the endpoints of e identified ("the contraction of e"). If v is a vertex of G we define $G - v$ to be the graph G with the vertex v and all edges incident with v removed. (We call the set of edges incident with a vertex v the *spine* of v and denote it by $\mathrm{sp}(v)$.) Then we have the following

3.1 Theorem (*cf.* [12], Theorem 2, p. 591). *For any graph G and every edge $e \in EG$ we have*

$$S_G(x,y) = S_{G-e}(x,y) + xyS_{G/e}(x,y) + xy(y-1)S_{(G/e)-w}(x,y)$$

if e is not a loop, and

$$S_G(x,y) = (1+x)S_{G-e}(x,y) + x(y-1)S_{(G/e)-w}(x,y)$$

if e is a loop. Here w is the vertex in G/e to which the endpoints of e have been identified.

Proof. Either e is contained in a given subset T or not. Hence we may write

$$S_G(x,y) = \sum_{e \notin T \subseteq G} x^{|T|} y^{|VT|} + \sum_{e \in T \subseteq G} x^{|T|} y^{|VT|}$$

[6] As is clear from (3.1), here and throughout the paper the word 'subgraph' means the graph whose vertices are VT and whose edges are T, where T is some subset of the edges of G. We could call them *edge induced* subgraphs, but this is too cumbersome.

[7] One can, however, derive the Tutte polynomial from a generalization of the subgraph polynomial (see [9] and [12]).

The first sum is clearly $S_{G-e}(x,y)$. We further divide the second sum into two subcases, depending on T. Let u and v be the endpoints of e in G, so that u and v are identified to w in G/e. There is an obvious one-to-one correspondence between subgraphs T of G containing e and subgraphs T' of G/e, but their weights are different depending on how $T-e$ meets u and v.

Case (1): $T-e$ meets u or v (or both). T' contains one fewer vertex and one fewer edge than T, so if we multiply the weight of T' by xy then it equals the weight of T in G. Note that in this case T' meets w, so it contains an edge of $\mathrm{sp}(w)$.

Case (2): $T-e$ meets neither u nor v. Then T' contains one fewer edge but two fewer vertices than T, so we must multiply the weight of T' by xy^2 to make it equal the weight of T in G. In this case T' does not meet w, so it does not contain any edge of $\mathrm{sp}(w)$.

Hence

$$\begin{aligned}\sum_{\substack{e\in T\subseteq G\\ (T-e)\cap\{u,v\}\neq\emptyset}} x^{|T|}y^{|VT|} &= xy \sum_{\substack{T'\subseteq G/e\\ T'\cap\mathrm{sp}(w)\neq\emptyset}} x^{|T'|}y^{|VT'|}\\ &= xy\sum_{T'\subseteq G/e} x^{|T'|}y^{|VT'|} - xy \sum_{\substack{T'\subseteq G/e\\ T'\cap\mathrm{sp}(w)=\emptyset}} x^{|T'|}y^{|VT'|}\end{aligned}$$

and

$$\sum_{\substack{e\in T\subseteq G\\ (T-e)\cap\{u,v\}=\emptyset}} x^{|T|}y^{|VT|} = xy^2 \sum_{\substack{T'\subseteq G/e\\ T'\cap\mathrm{sp}(w)=\emptyset}} x^{|T'|}y^{|VT'|}$$

Adding the two expressions gives the desired result. The formula in the case that e is a loop follows analogously. ∎

We also record the following easy fact:

3.2 PROPOSITION. *The subgraph polynomial satisfies*

$$S_{G\sqcup H}(x,y) = S_G(x,y)S_H(x,y)$$

where $\sqcup$ is disjoint union.

Proof. Every subset T of edges of $G\sqcup H$ is of the form (T',T'') where $T'\subseteq G$ and $T''\subseteq H$. The weight of T in $S_{G\sqcup H}$ is the product of the weights of T' and T''. ∎

4. The Hilbert Series of the Face Ring of a Flag Complex

The main result of this work is

4.1 Theorem. *Let G be a graph on n vertices and let $\Delta(G)$ be its clique complex. Then the Hilbert series of $k[\Delta(G)]$ is given by*

$$H_{k[\Delta(G)]}(t) = \frac{S_{\overline{G}}(-1,t)}{(1-t)^n}$$

where $S_{\overline{G}}(x,y)$ is the subgraph polynomial of $\overline{G}$.

Proof. [8] Define a finely graded analogue of the polynomial $S_M(t)$ appearing in Theorem 1.6, as follows:

$$\widetilde{S}_{k[\Delta(G)]}(t) := \sum_{i,\alpha} (-1)^i \beta_{i,\alpha}(k[\Delta(G)])t^\alpha \tag{4.1}$$

where, as before, $t = (t_1, \ldots, t_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $t^\alpha = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. Then from Theorem 2.1 we have (after a suitable shift of dummy indices)

$$\widetilde{S}_{k[\Delta(G)]}(t) = \sum_{\alpha} \widetilde{\chi}((0, x^\alpha))t^\alpha \tag{4.2}$$

where

$$\widetilde{\chi}(X) = \sum_i (-1)^i \dim \widetilde{H}_i(X) \tag{4.3}$$

is the reduced Euler characteristic of X. Observe that equation (2.1) and Theorem 1.6 imply

$$S_{k[\Delta(G)]}(t) = \widetilde{S}_{k[\Delta(G)]}(t, t, \ldots, t) \tag{4.4}$$

So, to compute $S_{k[\Delta(G)]}(t)$ we must compute the (reduced) Euler characteristics of the principal order ideals in the lcm-lattice corresponding to the ideal $I_{\Delta(G)}$.

To this end, label the vertices VG of G from 1 to n. To every edge of the form $e = \{i, j\}$ there is a unique monomial of degree 2, namely $x^e = x_i x_j$. We call x^e for some e an *edge monomial* of G.

Now, the minimal nonfaces of $\Delta(G)$ are precisely the edges of $\overline{G}$. Hence the monomial ideal $\overline{I} := I_{\Delta(G)}$ defining the face ring $k[\Delta(G)]$ is generated by the edge monomials of $\overline{G}$.

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[8] When this article was in preprint form several people pointed out to the author that this result may also be obtained by a direct inclusion-exclusion argument without recourse to the results of [1].

In particular, the lcm lattice $L := L_{\bar{I}}$ has the edge monomials of $\overline{G}$ as its atoms. The key point is that the least common multiple of a set of edge monomials corresponding to a subset T of edges of $\overline{G}$ is just the squarefree monomial whose support is VT, the vertices of T. Conversely, associated to every monomial $x^\alpha \in L$ there is a unique subgraph $\overline{G}_\alpha$ of $\overline{G}$, namely the induced subgraph on the support of α.

From Philip Hall's Theorem ([13], Proposition 3.8.6, p. 120) we have

$$\widetilde{\chi}((0, x^\alpha)) = \mu_L(0, x^\alpha) \tag{4.5}$$

the Möbius function of the closed interval $[0, x^\alpha]$ in L. As $[0, x^\alpha]$ is a sublattice of the lattice L, Rota's Cross Cut Theorem ([13], Corollary 3.9.4, p.125; [14], Theorem 4.42, p. 175) gives

$$\mu_L(0, x^\alpha) = \sum_k (-1)^k N(\alpha)_k \tag{4.6}$$

where $N(\alpha)_k$ is the number of k subsets of the atoms of L whose join is x^α. By the correspondence above between subgraphs of $\overline{G}$ and elements of L, a set of edge monomials corresponding to some subset T of edges has join x^α precisely when VT coincides with the support of α. In graph theoretical terminology, this occurs when T is a spanning subgraph of $\overline{G}_\alpha$. Hence $N(\alpha)_k$ counts the number of spanning subgraphs of $\overline{G}_\alpha$ with k edges.

Putting all this together with (4.2) gives

$$\widetilde{S}_{k[\Delta(G)]}(t) = \sum_{k,\alpha} (-1)^k N(\alpha)_k t^\alpha \tag{4.7}$$

Finally, setting $t_i = t$ for all i yields (from (4.4) and the fact that each spanning subgraph of $\overline{G}_\alpha$ occurs precisely once in $\overline{G}$)

$$S_{k[\Delta(G)]}(t) = \sum_{k,j} (-1)^k N_k t^j \tag{4.8}$$

where N_k is the number of subgraphs of $\overline{G}$ with k edges and $|\alpha| = j$ vertices. The result follows by comparing (4.8) and (3.2). ∎

4.2 Corollary. *Let G be a graph on n vertices and let $\overline{\Delta}(G)$ be its independent set complex. Then the Hilbert series of $k[\overline{\Delta}(G)]$ is given by*

$$H_{k[\overline{\Delta}(G)]}(t) = \frac{S_G(-1, t)}{(1-t)^n}$$

where $S_G(x, y)$ is the subgraph polynomial of G.

Proof. $\overline{\Delta}(G) = \Delta(\overline{G})$. ∎

Recall that a *vertex cover* of a graph G is a set V of vertices of G such that every edge is incident with at least one element of V.

4.3 COROLLARY. *Let $\beta(G)$ denote the number of vertices in a minimum vertex cover of G. Then*

$$\beta(G) = \min\{i : d^i S_G(-1,t)/dt^i \mid_{t=1} \neq 0\}$$

First Proof. Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k. Let $I \subseteq R$ be an ideal and denote by $Z(I)$ the affine variety (zero set) of I in k^n. The codimension of the ideal I is precisely the codimension of $Z(I)$ as a variety. When I is a monomial ideal $Z(I)$ is just a union of coordinate subspaces, and its dimension is just that of the maximal such subspace. This yields a simple recipe for computing the dimension. Let $I = \langle m_1, \ldots, m_s \rangle$ be the ideal generated by the monomials $\{m_1, \ldots, m_s\}$. Let M_j be the support of m_j for $1 \leq j \leq s$, let σ be any subset of $\{1, \ldots, n\}$ that has nonempty intersection with every M_j, $1 \leq j \leq s$, and let Σ be the set of all such σ. Then by Proposition 3 of ([15], p. 431) the codimension of I is $\min\{|\sigma| : \sigma \in \Sigma\}$.

Now, let G be a graph, $\Delta(G)$ its clique complex, and $k[\Delta(G)] = R/I_{\Delta(G)}$ its face ring. As before, $I_{\Delta(G)}$ is generated by all monomials of degree 2 corresponding to edges of $\overline{G}$. In this case the set Σ defined above corresponding to the ideal $I_{\Delta(G)}$ is precisely the set of vertex covers of $\overline{G}$. Hence the codimension of $I_{\Delta(G)}$ is just $\beta(\overline{G})$. The result now follows from (1.10) and Theorem 4.1. ∎

Second Proof. Let $\alpha(G)$ denote the size of a maximum independent set in G. The complement of a minimum vertex cover of G is a maximum independent set of G, so $\alpha(G)+\beta(G) = |VG|$. But $\dim k[\overline{\Delta}(G)] = 1 + \dim \overline{\Delta}(G) = \alpha(G)$, so the codimension of $I_{\overline{\Delta}(G)}$ is just $\beta(G)$. The result now follows from (1.10) and Corollary 4.2. ∎

4.4 COROLLARY. *Let G be a graph and $\Delta(G)$ its clique complex. Then the h-polynomial of $\Delta(G)$ is given by*

$$Q_{k[\Delta(G)]}(t) = \sum h_i t^i = \frac{S_{\overline{G}}(-1,t)}{(1-t)^{\beta(\overline{G})}}$$

Proof. Let d be the dimension of $I_{\Delta(G)}$. Then from the proof of Corollary 4.3, $\beta(\overline{G}) = \operatorname{codim} I_{\Delta(G)} = n - d$. The result now follows by equating the Hilbert series expressions in (1.6) and Theorem 4.1. ∎

5. Computational Considerations

Corollary 4.3 shows that computing the Hilbert series of the face ring of a flag complex (equivalently, the appropriately specialized subgraph polynomial) is a hard in a technical sense, as the problem of computing the minimum vertex cover of a graph is NP-complete ([16], p. 234 and Lemma 3.1.13, p. 104; see also [17], Proposition 2.9). Nevertheless, it is possible to improve upon the naive algorithm for $S_G(-1,t)$ derivable from Theorem 3.1 by appealing to a result of Bayer and Stillman [17]. We now recall their result, following Eisenbud ([8], p. 325).

Let $I \subseteq R$ be the ideal whose Hilbert series we wish to compute, and choose a minimal generator $m \in I$ of degree r. Set $I = (I', m)$ where $I' = (m_1, \ldots, m_s)$ is a monomial ideal generated by fewer monomials than I. We can construct an exact sequence of degree zero maps between graded modules

$$R(-r) \xrightarrow{\varphi} R/I' \longrightarrow R/I \longrightarrow 0 \tag{5.1}$$

where $R(-r)$ is the free module with generator in degree r and φ sends the generator of $R(-r)$ to the class of m in R/I'. The kernel of φ is the set of all elements f in R that satisfy $fm \in I'$, which is to say, the kernel of φ is the colon ideal $J := (I' : m)$, shifted in degree to be a submodule of $R(-r)$. One can show that

$$J = \left(\frac{m_1}{\gcd(m_1, m)}, \ldots, \frac{m_s}{\gcd(m_s, m)} \right) \tag{5.2}$$

so that J has fewer monomials than I.

From (5.1) we get the short exact sequence of graded modules

$$0 \longrightarrow (R/J)(-r) \longrightarrow R/I' \longrightarrow R/I \longrightarrow 0 \tag{5.3}$$

which gives, for each integer j, a short exact sequence of vector spaces

$$0 \longrightarrow (R/J)_{j-r} \longrightarrow (R/I')_j \longrightarrow (R/I)_j \longrightarrow 0 \tag{5.4}$$

As the Hilbert series is easily seen to be additive on exact sequences we conclude that

$$H_{R/I}(t) = H_{R/I'}(t) - t^r H_{R/J}(t) \tag{5.5}$$

We may cast the recursion (5.5) directly in terms of graphs. To avoid some notational confusion between G and its complement, we will temporarily use Γ to denote an arbitrary graph. Next, we define

$$H_\Gamma(t) := H_{k[\overline{\Delta}(\Gamma)]}(t) = H_{k[\Delta(\overline{\Gamma})]}(t) = H_{R/I}(t) \tag{5.6}$$

where I is the *edge ideal* of Γ consisting of all edge monomials of Γ. Let e be an edge of Γ and set $e = \{u, v\}$. Let $N(u)$ be the neighborhood of u, namely the set of vertices in Γ adjacent to u, and set $N(u, v) = N(u) \cup N(v)$. (Observe that $N(u, v)$ includes u and v.) The graph $\Gamma - e$ is Γ with e removed, while $\Gamma - N(u, v)$ means Γ with both the vertices $N(u, v)$ and all the edges incident with them removed.

We then have [9]

5.1 THEOREM. *For any simple graph* Γ

$$H_\Gamma(t) = H_{\Gamma-e}(t) - \left(\frac{t}{1-t}\right)^2 H_{\Gamma-N(u,v)}(t) \tag{5.7}$$

Proof. Let $m = x^e$, the edge monomial of e. The origin of the first term on the right hand side of (5.7) is clear, as the monomial ideal I' in (5.5) consists of all the edges of Γ except e. The second term in (5.7) derives from the colon ideal J. Examining (5.2) we see that J consists of degree one monomials corresponding to the vertices in $N(u, v)$ (except u and v) and degree two monomials corresponding to edges not incident with $N(u, v)$. Let J' be the ideal generated by J and the monomials x^u and x^v. Using (5.5) gives $H_{R/J'}(t) = (1-t)^2 H_{R/J}(t)$. But $H_{R/J'}(t) = H_{\Gamma-N(u,v)}(t)$. ∎

5.2 EXAMPLE. Let Γ be the graph on 8 vertices with the following edges: $\{1, 3\}$, $\{1, 4\}$, $\{5, 7\}$, $\{5, 8\}$, $\{1, 5\}$, $\{2, 6\}$, $\{3, 7\}$, and $\{4, 8\}$. Then

$$I = (x_1x_3, x_1x_4, x_5x_7, x_5x_8, x_1x_5, x_2x_6, x_3x_7, x_4x_8) \tag{5.8}$$

If we pick, say, $m = x_1x_3$, then

$$I' = (x_1x_4, x_5x_7, x_5x_8, x_1x_5, x_2x_6, x_3x_7, x_4x_8) \tag{5.9}$$

which clearly has fewer generators than I. Also, from (5.2) we have

$$J = (x_4, x_5x_7, x_5x_8, x_5, x_2x_6, x_7, x_4x_8) = (x_4, x_5, x_7, x_2x_6) \tag{5.10}$$

9 A recursion similar to the one in Theorem 5.1 was found by Watkins in his unpublished master's thesis ([18], Theorem 5.1; see also [19], p. 21). (I am grateful to Wolmer Vasconcelos for providing me with a copy of Watkins' thesis.) Written in our notation it reads

$$H_\Gamma(t) = H_{\Gamma-u}(t) + \left(\frac{t}{1-t}\right) H_{\Gamma-u-N(u)}(t)$$

where u is an arbitrary vertex of Γ, $N(u)$ is its neighborhood, and, as above, removing a vertex also entails removing its incident edges.

which consists of degree one monomials corresponding to vertices of Γ incident with the edge $\{1,3\}$ and degree two monomials corresponding to edges of Γ not incident with vertices 1 and 3.

Proceeding in this way we may compute the Hilbert series of $k[\Delta(\overline{\Gamma})]$, which is [10]

$$H_{k[\Delta(\overline{\Gamma})]}(t) = \frac{1-8t^2+10t^3+4t^4-12t^5+4t^6+2t^7-t^8}{(1-t)^8}$$

We also find the codimension of the corresponding ideal: $\mathrm{codim} I_{\Delta(\overline{\Gamma})} = 4$.

To compare this result to Corollary 4.2 we must compute the subgraph polynomial of Γ, which we do separately for each connected component:

$$\begin{aligned} S_{\Gamma-\{2,6\}}(x,y) = 1 &+ 7xy^2 + 10x^2y^3 + 11x^2y^4 + 16x^3y^4 + 16x^3y^5 + 3x^3y^6 \\ &+ 2x^4y^4 + 18x^4y^5 + 15x^4y^6 + 4x^5y^5 + 17x^5y^6 + 7x^6y^6 + x^7y^6 \end{aligned}$$

and

$$S_{\{2,6\}}(x,y) = 1 + xy^2$$

Putting these results together using Proposition 3.2 we get

$$\begin{aligned} S_\Gamma &= S_{\Gamma-\{2,6\}}(x,y)S_{\{2,6\}}(x,y) \\ &= 1 + 8xy^2 + 10x^2y^3 + 18x^2y^4 + 16x^3y^4 + 26x^3y^5 + 2x^4y^4 + 14x^3y^6 + 18x^4y^5 \\ &\quad + 31x^4y^6 + 4x^5y^5 + 16x^4y^7 + 19x^5y^6 + 3x^4y^8 + 18x^5y^7 + 7x^6y^6 \\ &\quad + 15x^5y^8 + 4x^6y^7 + x^7y^6 + 17x^6y^8 + 7x^7y^8 + x^8y^8 \end{aligned}$$

Specializing gives

$$S_\Gamma(-1,t) = 1-8t^2+10t^3+4t^4-12t^5+4t^6+2t^7-t^8$$

which is precisely the numerator of the Hilbert series. Dividing by $(1-t)^4$ yields the $h-$polynomial

$$Q_{k[\Delta(\overline{\Gamma})]}(t) = 1 + 4t + 2t^2 - 2t^3 - t^4$$

and from the $h-$polynomial we arrive at the $f-$polynomial of $\Delta(\overline{\Gamma})$ using (1.8):

$$\sum_i f_i t^i = 1 + 8t + 20t^2 + 18t^3 + 4t^4$$

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[10] This result was verified using the computer program `Macaulay` created by Dave Bayer and Mike Stillman (which employs the algorithm (5.5)).

References

1) V. Gasharov, I. Peeva, and V. Welker, "The LCM-Lattice", *Math. Res. Lett.* **6** (1999) 521-532.

2) J. Tits, *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics, No. 386 (Springer, New York, 1974).

3) R. Stanley, *Combinatorics and Commutative Algebra*, 2^{nd} ed. (Birkhauser, Boston, 1996).

4) R. Charney and M. Davis, "The Euler Characteristic of a Nonpositively Curved, Piecewise Euclidean Manifold", *Pac. J. Math.* **171** (1995) 117-137.

5) T. Hibi, *Algebraic Combinatorics on Convex Polytopes* (Carslaw Publications, Glebe, Australia, 1992).

6) W. Bruns and J. Herzog, *Cohen-Macaulay Rings* (Cambridge University Press, Cambridge, 1993).

7) V. Gasharov, I. Peeva, and V. Welker, "Coordinate Subspace Arrangements and Monomial Ideals", to appear in the proceedings of the meeting on Computational Commutative Algebra and Combinatorics, July 21-30, 1999, Osaka, Japan. Preprint available from http://www.mathematik.uni-marburg.de/ ˜welker/sections/preprints.html

8) D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry* (Springer, New York, 1995).

9) E. Farrell, "On a General Class of Graph Polynomials", *J. Combin. Theory Ser. B* **26** (1979) 111-122.

10) L. Borzacchini, "Subgraph Enumerating Polynomial and Reconstruction Conjecture", *Rend. Accad. Sci. Fis. Mat. Napoli.* **43** (1976) 411-6.

11) T. Brylawskyi, "A Decomposition for Combinatorial Geometries", *Trans. Amer. Math. Soc.* **171** (1972) 235-282.

12) L Borzacchini and C. Pulito, "On Subgraph Enumerating Polynomials and Tutte Polynomials", *Boll. Un. Mat. Ital. B* **1** (1982) 589-597.

13) R. Stanley, *Enumerative Combinatorics*, Vol. 1 (Wadsworth & Brooks/Cole, Monterey, CA, 1986).

14) M. Aigner, *Combinatorial Theory* (Springer-Verlag, Berlin, 1997).

15) D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, 2^{nd} ed. (Springer, New York, 1997).

16) D. West, *Introduction to Graph Theory* (Prentice Hall, Upper Saddle River, NJ, 1996).

17) D. Bayer and M. Stillman, "Computation of Hilbert Functions", *J. Symbolic Comp.* **14** (1992) 31-50.

18) A. Watkins, *Hilbert Functions of Face Rings Arising From Graphs* (M.S. Thesis, Rutgers University, 1992).

19) W. Vasconcelos, *Computational Methods in Commutative Algebra and Algebraic Geometry* (Springer-Verlag, Berlin, 1998).