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June 13, 2014

Outline

1. Reflection/Coxeter Groups
2. Reflection Arrangements
3. Finite Field Method

Root Systems and Reflection Groups

- $V$ an $n$ dimensional inner product space over $\mathbb{R}$
- Given $\alpha \in V$, reflection $t_\alpha : V \to V$ fixes hyperplane $H_\alpha := \{v \in V | (\alpha, v) = 0\}$ (pointwise) and sends $\alpha$ to $-\alpha$.
- $\Phi \subset V$ is a root system if
  - $t_\alpha \Phi = \Phi$, $\alpha \in \Phi$, and
  - $\Phi \cap \mathbb{R} \alpha = \{-\alpha, \alpha\}$ for all $\alpha \in \Phi$.
- Let $W(\Phi)$ be the group generated by reflections $t_\alpha$, $\alpha \in \Phi$
- If $W$ is finite it is called a finite reflection group or finite Coxeter group
- $W$ is a Weyl group if $\Phi$ is crystallographic: $2(\alpha, \beta)/(\beta, \beta) \in \mathbb{Z}$
Simple Systems

- \( \Delta \subseteq \Phi \) is a simple system provided
  - \( \Delta \) is a basis for \( V \).
  - Every root \( \alpha \in \Phi \) can be written \( \alpha = \sum_{\alpha_i \in \Delta} c_i \alpha_i \) where all \( c_i \geq 0 \) or all \( c_i \leq 0 \).
- The elements of a simple system are called simple roots.
- The positive roots \( \Phi^+ \) are those for which \( c_i > 0 \) for all \( i \).
- In crystallographic case, \( c_i \in \mathbb{Z} \) for all \( i \).

Generators and Relations

- Fix a simple system \( \Delta \in \Phi \).
- Write \( s_\alpha \) instead of \( t_\alpha \) whenever \( \alpha \in \Delta \). (The simple reflections.)

**Theorem (The Coxeter Presentation)**

\( W \) is generated by the set of simple reflections subject only to the relations

\[ (s_\alpha s_\beta)^{m(\alpha, \beta)} = 1 \quad (\alpha, \beta \in \Delta). \]

- The minimum length of a word \( w \) written in terms of simple (respectively, all) reflections is called the length (respectively, absolute length) of \( w \).

The Classification

The Coxeter graph of \( W \) is the graph with one vertex for each simple reflection and with edges \((s_\alpha, s_\beta)\) labeled by the integers \( m(\alpha, \beta) \). Edges labeled by 2 are suppressed.

\[ \begin{align*}
\ldots & \quad \ldots \\
A_n & \quad B_n = C_n & \quad E_7 & \quad F_4 \\
D_n & \quad E_6 & \quad G_2 & \quad H_3 \\
 & \quad G_2 & \quad H_2 \quad n & \quad H_4 \\
& \quad H_2 & \quad 5 & \quad 5 & \quad 5
\end{align*} \]

\( W \) is irreducible if the Coxeter graph is connected. All Coxeter groups are direct products of irreducible ones.
**Coxeter Transformations**

- The **Coxeter element** of $W$ is $c = \prod_{s \in S} s$ (unique up to conjugacy)
- The order of $c$ is $h$, the **Coxeter number**
- The eigenvalues of $c$ (in the reflection representation) are of the form $\zeta^{m_i}$ for some primitive $h^{th}$ root of unity $\zeta$.
- The numbers $m_1, m_2, \ldots, m_n$ are the **exponents** of $W$.

**The Exponents of Reflection Groups**

<table>
<thead>
<tr>
<th>Type</th>
<th>$m_1, m_2, \ldots, m_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$1, 2, \ldots, n$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$1, 3, 5, \ldots, 2n - 1$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$1, 3, 5, \ldots, 2n - 1, n - 1$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$1, 4, 5, 7, 8, 11$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$1, 5, 7, 9, 11, 13, 17$</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$1, 7, 11, 13, 17, 19, 23, 29$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$1, 5, 7, 11$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$1, 5$</td>
</tr>
<tr>
<td>$H_3$</td>
<td>$1, 5, 9$</td>
</tr>
<tr>
<td>$H_4$</td>
<td>$1, 11, 19, 29$</td>
</tr>
<tr>
<td>$I_2(m)$</td>
<td>$1, m - 1$</td>
</tr>
</tbody>
</table>

**Exponents in Surprising Places**

**Theorem (Chevalley '55, Solomon '66, Steinberg '68)**

Let $\ell_S(w)$ denote the minimum length of $w$ as a word in the simple reflections. Then

$$\sum_{w \in W} q^{\ell_S(w)} = \prod_{i=1}^{n} \frac{1 - q^{m_i+1}}{1 - q}$$

**Theorem (Shephard & Todd '54, Solomon '63)**

Let $\ell_T(w)$ denote the minimum length of $w$ as a word in all the reflections. Then

$$\sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{n} (1 + m_i q)$$

**Theorem (Brieskorn '71)**

Let $M = V_{\mathbb{C}} \setminus \bigcup H_\alpha$. Then

$$\sum_{i \geq 0} \dim H^i(M, \mathbb{C}) t^i = \prod_{i=1}^{n} (1 + m_i t)$$

And the list goes on.... Exponents appear in many beautiful formulas for nonnesting partitions, noncrossing partitions, cluster algebras, etc.
Main Question

Shephard and Todd proved their result case-by-case for unitary reflection groups (a generalization of real reflection groups). The other proofs all used invariant theory. (Brieskorn reduced his problem to the Shephard-Todd result.)

Can one find a simple combinatorial explanation for the exponents?

Normal Forms for Reflection Group Elements

- Look again at the Shephard-Todd formula
  \[ \sum_{w \in W} q^{\ell_T(w)} = \prod_{i=1}^{n} (1 + m_i q). \]
- Wouldn’t it be nice if this were the shadow of something deeper?

**Theorem (Known?)**

Let \( W \) be of type \( A_n \) or \( B_n \) and let \( T \) be the set of all reflections of \( W \). Then there exists a partition of \( T \) into classes \( T_i \) with \(|T_i| = m_i\) satisfying
  \[ \sum_{w \in W} q^{\ell_T(w)} w = \prod_{i=1}^{n} (1 + qX_i) \]
  as an identity in the group algebra \( \mathbb{R}[q]W \), where \( X_i := \sum_{t \in T_i} t \).

Proof

Hyperplane Arrangements and the Lattice of Flats

- A collection of hyperplanes is called a hyperplane arrangement.
- The intersection of any subcollection of hyperplanes is called a flat.
- To every arrangement \( \mathcal{A} \) we associate the intersection poset whose elements are the flats, ordered by reverse inclusion so that \( A \leq B \iff A \supseteq B \).
- The arrangement is central if all the hyperplanes intersect in a point, in which case the intersection poset is a lattice (joins and meets exist), called the lattice of flats.
The $W$-Partition Lattice

- A reflection arrangement is the set of all reflecting hyperplanes of a reflection group $W$.
- The lattice of flats of a reflection arrangement $L_W$ is called the $W$-partition lattice.

The $W(A_2)$ Partition Lattice–cont.

The Möbius Function and the Characteristic Polynomial

- The Möbius function $\mu(x, y)$ is defined recursively. We only need $\mu(\hat{0}, x)$, which is computed as follows.
  1. $\mu(x, x) = 1$.
  2. $\mu(\hat{0}, x) = -\sum_{y < x} \mu(\hat{0}, y)$
- The characteristic polynomial of $L$ is
  \[ \chi(A, q) = \sum_{x \in L} \mu(\hat{0}, x)q^{\dim(x)}. \]

Example

\[ \chi(A, q) = \sum_{x \in L} \mu(\hat{0}, x)q^{\dim(x)} = q^2 - 3q + 2 \]
Reflection/Coxeter Groups

Reflection Arrangements

Finite Field Method

Characteristic Polynomials of Reflection Arrangements

Why is all this relevant?

Theorem (Orlik and Solomon ‘80)

Let $L$ be the partition lattice associated to the reflection group $W$. Then

$$
\chi(L, q) = \prod_{i=1}^{n} (q - m_i),
$$

where the $m_i$ are the exponents of $W$.

But how to compute $\chi(L, q)$?

Finite Field Method

- Recall that the defining equation of a hyperplane can be written $a_1 x_1 + \cdots + a_n x_n = b$ for some real numbers $\{a_1, a_2, \ldots, a_n, b\}$.
- In many cases of interest the numbers $a_1, a_2, \ldots, a_n, b$ are integers (an integral arrangement).
- When this holds there is a particularly nice way to compute the characteristic polynomial.
- For any positive integer $q$ let $A_q$ denote the hyperplane arrangement $A$ with defining equations reduced mod $q$.

Remarks

- We need large $q$ to avoid lowering the dimension of any of the flats by accident.
- But two polynomials that agree for enough values of $q$ are equal.
- Identifying $F_q^n$ with $\{0, 1, \ldots, q - 1\}^n = [0, q - 1]^n$, $\chi(A, q)$ is the number of points in $[0, q - 1]^n$ that do not satisfy modulo $q$ the defining equations of any of the hyperplanes in $A$. 

The Characteristic Polynomial for Integral Arrangements

Theorem (Crapo and Rota ’71, Orlik and Terao ’92, Blass and Sagan ’96, Athanasiadis ’96, Björner and Ekedahl ’96)

For sufficiently large primes $q$,

$$
\chi(A, q) = \# \left( F_q^n - \bigcup_{H \in A_q} H \right),
$$

where $F_q^n$ denotes the vector space of dimension $n$ over the finite field with $q$ elements.
Weyl Arrangements

- Weyl arrangements are integral, so method applies.
- Three infinite families associated to types $A_{n-1}$, $D_n$, and $B_n$, respectively:
  
  $A_{n-1} = \{ x_i - x_j = 0 \mid 1 \leq i \leq j \leq n \}$
  
  $D_n = A_n \cup \{ x_i + x_j = 0 \mid 1 \leq i \leq j \leq n \}$
  
  $B_n = D_n \cup \{ x_i = 0 \mid 1 \leq i \leq n \}$

Computing the Characteristic Polynomial I

- What is $\chi(A_{n-1})$?
- According to the finite field method, we want the number of points in $[0,q-1]^n$ satisfying $x_i \neq x_j$ for all $1 \leq i \leq j \leq n$.
- This is the same thing as asking for vectors $(x_1,x_2,\ldots,x_n)$ all of whose entries are distinct mod $q$.
- Well, we can pick $x_1$ in $q$ ways, then $x_2$ in $q-1$ ways, and so on. Thus
  
  $\chi(A_{n-1},q) = q(q-1)(q-2)\cdots(q-n+1).$

Computing the Characteristic Polynomial II

- What is $\chi(B_n)$?
- Now we want to count the points satisfying $x_i \neq x_j$, $x_i \neq -x_j$, and $x_i \neq 0$.
- Since we do not allow 0, there are only $q-1$ (nonzero) choices for the first entry, $q-3$ nonzero choices for the second entry (because we must avoid the first entry and its negative), etc..
- Thus
  
  $\chi(B_n,q) = (q-1)(q-3)\cdots(q-2n+1).$

Computing the Characteristic Polynomial III

- What is $\chi(D_n)$?
- Now we want to count the points satisfying $x_i \neq x_j$, and $x_i \neq -x_j$, but this time we allow 0 as an entry.
- There are two possibilities. If zero is not present, then $B_n$ case. If zero is present, then $B_{n-1}$ case for remaining entries.
There are $n$ choices for the placement of the zero, so we get
\[
\chi(D_n, q) = (q - 1)(q - 3) \cdots (q - 2n + 1)
+ (q - 1)(q - 3) \cdots (q - 2n + 3)(n)
= (q - 1)(q - 3) \cdots (q - 2n + 3)(q - n + 1)
\]

Basic reason for failure of normal form theorem in type $D_n$ (as well as other types).

Is there another way?

Instead of $F_n^q$ use a symmetry adapted lattice.

Let $\Phi$ be crystallographic with simple roots $\{\alpha_i\}$.

Recall $c_\alpha(\alpha)$ given by $\alpha = \sum_i c_\alpha(\alpha)i$.

There exists a unique highest root $\tilde{\alpha}$ with largest coefficients, denoted $\tilde{c}_i$.

**Theorem** (Haiman '93, Blass and Sagan '96, Athanasiadis '96, Terao et al '07)

Let $\mathcal{A}$ be the Weyl arrangement associated to $\Phi$, and let $\sigma$ be the rational simplex in the nonnegative orthant of $\mathbb{Z}^n$ bounded by the hyperplane $\sum_i \tilde{c}_i x_i = 1$. Let $M := \text{lcm}(\tilde{c}_1, \ldots, \tilde{c}_n)$. Then for $q$ relatively prime to $M$, $\chi(\mathcal{A}, q) = C\bar{\chi}(\sigma, q)$, where $C := n! \prod_i \tilde{c}_i$.

**Corollary**

Let $W$ be a Weyl group with Coxeter number $h$, reflection arrangement $\mathcal{A}$, and minimum quasiperiod $M$. Then the characteristic polynomial $\chi(\mathcal{A}, q)$ vanishes (and therefore $q$ is an exponent) whenever $q \leq h - 1$ and $(q, M) = 1$.

**Proof** (Was already known that $q$ is an exponent if $q \leq h - 1$ is coprime to $h$.)
### Exponents from Counting–cont.

<table>
<thead>
<tr>
<th>Type</th>
<th>Coefficients of $\tilde{\alpha}$</th>
<th>M</th>
<th>$h$</th>
<th>zeros of $\chi$ from cor.</th>
<th>missing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>1,1,..,1</td>
<td>1</td>
<td>$n+1$</td>
<td>1,2,..,n</td>
<td></td>
</tr>
<tr>
<td>$B_n$</td>
<td>1,2,..,2</td>
<td>2</td>
<td>$2n$</td>
<td>1,3,5,..,2n-1</td>
<td></td>
</tr>
<tr>
<td>$D_n$</td>
<td>1,1,1,..,2</td>
<td>2</td>
<td>$2n-2$</td>
<td>1,3,5,..,2n-3</td>
<td>4,8</td>
</tr>
<tr>
<td>$E_6$</td>
<td>1,1,2,2,2,3</td>
<td>6</td>
<td>12</td>
<td>1,5,7,11</td>
<td>9</td>
</tr>
<tr>
<td>$E_7$</td>
<td>1,2,2,2,3,3,4</td>
<td>12</td>
<td>18</td>
<td>1,5,7,11,13,17</td>
<td></td>
</tr>
<tr>
<td>$E_8$</td>
<td>2,2,3,3,4,4,5,6</td>
<td>60</td>
<td>30</td>
<td>1,7,11,13,17,19,23,29</td>
<td></td>
</tr>
<tr>
<td>$F_4$</td>
<td>2,2,3,4</td>
<td>12</td>
<td>12</td>
<td>1,5,7,11</td>
<td></td>
</tr>
<tr>
<td>$G_2$</td>
<td>2,3</td>
<td>6</td>
<td>6</td>
<td>1,5</td>
<td></td>
</tr>
</tbody>
</table>

### Work in Progress

- Can get rest of exponents from generating function, but not pretty.
- How is this related to theorem of Shapiro (?), Kostant ('59), Macdonald ('72) concerning root height partitions?
- Is there a counting interpretation of the exponents in the noncrystallographic case?
Roots of the Characteristic Polynomial

- $\bar{\iota}(s, q)$ counts all $x$'s with $x_i > 0$ and $\sum_i \tilde{c}_i x_i < q$.
- The smallest possible point in interior of $\sigma$ is $(1, 1, \ldots, 1)$.
- $\sum_i \tilde{c}_i = h - 1$ ($h$ the Coxeter number).
- Therefore $\bar{\iota}(s, q) = 0$ for $q \leq h - 1$.
- But $\chi(A, q) = C\bar{\iota}(\sigma, q)$ for $q$ prime to the quasiperiod.
- So $\chi(A, q) = 0$ for $q \leq h - 1$ and $(q, M) = 1$. 

Return