The Spectrum of the Derangement Graph

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May 3, 2007

Outline

Cayley Graphs

- G a finite group
- $S \subseteq G$ a symmetric subset of generators:

$$\{s \in G : s \in S \Longrightarrow s^{-1} \in S\}$$

- $\Gamma(G, S)$ a Cayley graph:

$$V(\Gamma) = G$$
$$E(\Gamma) = \{u \sim v \Leftrightarrow vu^{-1} \in S\}$$

- $\Gamma(G, S)$ is normal if S is closed under conjugation.

The Derangement Graph

- $G = S_n$ the symmetric group on $X = \{1, 2, \ldots, n\}$
- $S = \mathcal{D}_n$ the set of derangements (fixed point free permutations) on X:

$$\{\sigma \in S_n : \sigma(x) \neq x, \forall x \in X\}$$

- $\Gamma_n := \Gamma(S_n, \mathcal{D}_n)$ the derangement graph

Properties of the Derangement Graph

- Γ_n is connected for $n > 3$. S_n generated by adjacent transpositions, and $(k, k+1)$ can be written as $(1, 2, \ldots, n)^2 \cdot (n, n-1, \ldots, 1)^2 (k, k+1)$
- Γ_n is Hamiltonian. (Eggleton and Wallace, 1985)
- $\alpha(\Gamma_n) = (n-1)!$. (Deza and Frankl, 1977). Bound achieved by coset of stabilizer of a point. These are the only such maximum independent sets (Cameron and Ku, 2003).
- $\omega(\Gamma_n) = n$. Latin squares!
- $\chi(\Gamma_n) = n$. A normal Cayley graph with $\alpha\omega = |V|$ satisfies $\omega = \chi$ (Godsil, unpublished).

Delsarte-Hoffman Bound

Given any regular graph of degree k with N vertices we have the Delsarte-Hoffman bound

$$\alpha \leq \frac{N}{1 - k/\lambda}$$

where λ is the least eigenvalue of the (adjacency matrix) of the graph.
For the derangement graph $N = n!$, $\alpha = (n-1)!$, and $k = D_n = |\mathcal{D}_n|$ so

$$\lambda \geq \frac{-D_n}{n-1}$$

Conjecture (C. Ku)

Equality holds.

Equality would imply the Shannon capacity of Γ_n is n.

The Eigenvalues of a Normal Cayley Graph

Theorem (Diaconis and Shahshahani, 1981; Babai, 1974)

Let Γ be a normal Cayley graph with adjacency matrix A. Then the eigenvalues of A are given by

$$\eta_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s)$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of η_χ is $\chi(1)^2$.

Proof.

$$A_{\sigma\tau} = \begin{cases} 1 & \text{if } \sigma = s\tau \text{ for some } s \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Proof continued...

Consider

$$C := \sum_{s \in S} s \in \mathbb{C}[G]$$

As a linear operator on $\mathbb{C}[G]$

$$C \cdot \tau = \sum_{s \in S} s\tau = \sum_{\substack{\sigma \in G \\ \sigma = s\tau}} \sigma = \sum_{\sigma \in G} A_{\sigma\tau}\sigma$$

So eigenvalues of A are eigenvalues of C.

Proof completed

By normality, C is in the center of $\mathbb{C}[G]$, so C is a $\mathbb{C}[G]$ module endomorphism. By Schur's Lemma, C is a constant, say c, on any simple $\mathbb{C}[G]$ module V. Let ρ be the irreducible representation afforded by V and χ its character. Then $\rho(C) = cI$ where $c = \chi(C)/\chi(1) = \sum_{s \in S} \chi(s)/\chi(1)$. As the regular representation decomposes into a direct sum of simple modules with multiplicity equal to dimension, the result follows. □

Integrality of Derangement Graph Spectrum

Corollary

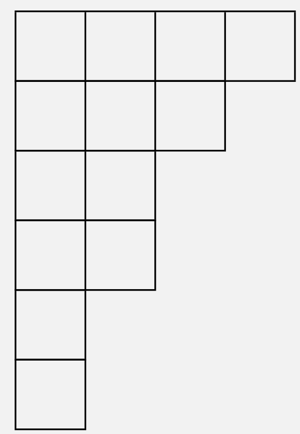
The eigenvalues of the derangement graph are integers.

Integer Partitions

Recall that a partition λ of n, written $\lambda \vdash n$ or $|\lambda| = n$, is a weakly decreasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\sum_i \lambda_i = n$. Its length is l and each λ_i is a part of the partition.

Partitions are represented by Ferrers diagrams:

$$(4, 3, 2, 2, 1, 1) \qquad \longleftrightarrow$$



and by multiplicity notation:

$$(4, 3, 2, 2, 1, 1) \qquad \longleftrightarrow \qquad 4^1 3^1 2^2 1^2$$

The Irreducible Characters of S_n

- To every permutation σ we associate a partition $\nu(\sigma)$, namely its cycle type.
- Conjugation preserves cycle type, so the conjugacy classes of S_n are in bijection with partitions of n.
- Hence the irreducible characters χ_λ of S_n are also in bijection with partitions of n.
- There exist algorithms to compute the irreducible characters of S_n, but there is no simple formula, in general.
- There are formulae for specific characters.

The Standard Representation of S_n

Let $V = \{e_1, e_2, \ldots, e_n\}$ be the defining representation of S_n:

$$\sigma(e_i) = e_{\sigma(i)}$$

S_n leaves fixed the one dimensional subspace U generated by the vector

$$e_1 + e_2 + \cdots + e_n$$

so U affords the trivial representation (which is clearly irreducible). It turns out that the orthogonal complement $W = U^{\perp}$ also affords an irreducible representation (of dimension $n-1$) called the standard representation of S_n. Thus we have the equivariant decomposition

$$V = U \oplus W$$

The Standard Character of S_n

By the properties of characters,

$$\chi_V = \chi_U + \chi_W$$

A moment's thought shows that

$$\chi_V(\sigma) = \#\text{ fixed points of } \sigma$$

Hence

$$\chi_W(\sigma) = \#\text{ fixed points of } \sigma - 1$$

The eigenvalue of Γ_n corresponding to this representation is thus

$$\eta_W = \frac{1}{\chi_W(1)} \sum_{s \in \mathcal{D}_n} \chi_W(s) = \frac{-D_n}{n-1}$$

This is the conjectured least eigenvalue!

Class	1^6	$2^1 1^4$	$2^2 1^2$	2^3	$3^1 1^3$	$3^1 2^1 1^1$	3^2	$4^1 1^2$	$4^1 2^1$	$5^1 1^1$	6^1	η_λ
# Elts	1	15	45	15	40	120	40	90	90	144	120	
6^1	1	1	1	1	1	1	1	1	1	1	1	+265
$5^1 1^1$	5	3	1	-1	2	0	-1	1	-1	0	-1	-53
$4^1 2^1$	9	3	1	3	0	0	0	-1	1	-1	0	+15
$4^1 1^2$	10	2	-2	-2	1	-1	1	0	0	0	1	+13
3^2	5	1	1	-3	-1	1	2	-1	-1	0	0	-11
$3^1 2^1 1^1$	16	0	0	0	-2	0	-2	0	0	1	0	-5
$3^1 1^3$	10	-2	-2	2	1	1	1	0	0	0	-1	-5
2^3	5	-1	1	3	-1	-1	2	1	-1	0	0	+7
$2^2 1^2$	9	-3	1	-3	0	0	0	1	1	-1	0	+5
$2^1 1^4$	5	-3	1	1	2	0	-1	-1	-1	0	1	+1
1^6	1	-1	1	-1	1	-1	1	-1	1	1	-1	-5

The Ring of Symmetric Functions

- S_n acts on elements of $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting indices.
- The ring of symmetric functions in n variables is

$$\Lambda_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$$

- Λ_n admits a natural grading into homogeneous pieces of degree k.
- There are many bases for Λ_n. We need two: the homogeneous symmetric functions and the Schur functions.

Complete Homogeneous Symmetric Functions

The homogeneous symmetric functions are

$$h_{\{\lambda_1,\lambda_2,\ldots,\lambda_l\}} = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$$

where h_k is the sum of all monomials of degree k. For example ($n = 3$)

$$\begin{aligned} h_1 &= x_1 + x_2 + x_3 \\ h_2 &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3 \end{aligned}$$

so

$$\begin{aligned} h_{(2,1)} &= h_2h_1 \\ &= x_1^3 + x_2^3 + x_3^3 + 2x_1^2x_2 + 2x_1^2x_3 \\ &\quad +2x_2^2x_3 + 2x_1x_2^2 + 2x_1x_3^2 + 2x_2x_3^2 \\ &\quad +3x_1x_2x_3 \end{aligned}$$

Young Tableaux

The Schur functions s_λ can be defined combinatorially. To every partition λ associate a semistandard Young tableau T (SSYT T: weakly increasing rows, strictly increasing columns) of shape λ

$$(4,2,1) \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

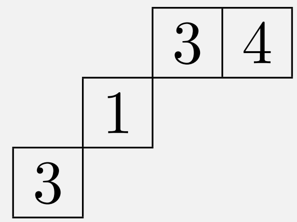
The type of T is a vector giving the multiplicities of each entry. In the above example, type $T = (2,3,1,1)$. Associated to a tableau is the monomial x^{type}. In the example,

$$x^T := x_1^2 x_2^3 x_3 x_4$$

(Skew) Schur Functions

Generalize. Let $\mu \subseteq \lambda$ (boxwise at upper left corner). Define a skew SSYT of shape λ/μ by removing the boxes in μ and filling in what remains

$$(4,2,1)/(2,1) \qquad \longleftrightarrow$$



The tableau monomial x^T is defined as before. The skew Schur function of shape λ/μ is

$$s_{\lambda/\mu} = \sum_T x^T$$

where the sum is over all skew SSYT of shape λ/μ. If $\mu = \emptyset$ then s_λ is the Schur function of shape λ

Hall Inner Product and Kostka Numbers

Define the canonical (Hall) inner product on symmetric functions

$$(s_\lambda, s_\mu) = \delta_{\lambda,\mu}$$

It turns out that

$$(s_\lambda, h_\mu) = K_{\lambda,\mu}$$

where $K_{\lambda,\mu}$ is the Kostka number, namely the number of semistandard Young tableau of shape λ and type μ.

Stanley's Theorem

Following Stanley, we define

$$d_\lambda = \sum_{s \in \mathcal{D}_n} \chi_\lambda(s)$$

Theorem (Stanley, EC2)

$$\sum_{\lambda \vdash n} d_\lambda s_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k h_{k1^{n-k}}$$

where $(n)_k = n!/(n-k)!$ *and the partition* $k1^{n-k}$ *means* k *followed by* $n-k$ *ones.*

Proof. Follows from Cauchy identity and Munaghan-Nakayama rule:

$$s_\lambda = \sum_{\sigma \in S_n} \chi_\lambda(\sigma) p_{\nu(\sigma)} \quad \square$$

The Eigenvalues of Derangement Graph

Theorem

The eigenvalues of the derangement graph are given by

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k \frac{f^{\lambda/k}}{f^\lambda}$$

Proof. Taking inner product with s_λ gives

$$d_\lambda = \sum_{k=0}^{n} (-1)^{n-k} (n)_k K_{\lambda, k1^{n-k}}$$

But

$$K_{\lambda, k1^{n-k}} = f^{\lambda/k}$$

where $f^{\lambda/\mu}$ is the number of SYT of skew shape λ/μ (SYT = strictly increasing in rows and columns)

Proof completed

Example. $n = 7$, $\lambda = (4, 2, 1)$, $k = 2$:

$(4, 2, 1)/(2) \quad \longleftrightarrow$

1	1	*	*
*	*		
*			

Finally, use theorem on eigenvalues of Cayley graph

$$\eta_\lambda = \frac{1}{\chi_\lambda(1)} d_\lambda$$

and fact that

$$\chi_\lambda(1) = f^\lambda$$

□

A More Explicit Form

Define the shifted partition

$$\mu_i := \lambda_i + l - i$$

Also, define

$$A(\mu) := \begin{vmatrix} (\mu_1)_{l+k-1} & \mu_1^{l-2} & \mu_1^{l-3} & \cdots & 1 \\ (\mu_2)_{l+k-1} & \mu_2^{l-2} & \mu_2^{l-3} & \cdots & 1 \\ (\mu_3)_{l+k-1} & \mu_3^{l-2} & \mu_3^{l-3} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}$$

and

$$\omega_k(\mu_1, \mu_2, \ldots, \mu_l) := \frac{A(\mu)}{\prod_{1 \le i \le j \le l} (\mu_i - \mu_j)}$$

Another Expression for the Eigenvalues

Theorem

The eigenvalues of the derangement graph are given by

$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} \omega_k(\mu_1, \mu_2, \ldots, \mu_l)$$

Proof. The Frobenius formula and the hook formula. □

Complete Factorial Symmetric Functions

Chen and Louck (1993) defined the complete factorial symmetric functions by

$$w_k(z_1, z_2, \ldots, z_n) = \sum_{i_1+i_2+\cdots+i_n=k} \prod_{1\leq j\leq n} (z_j - i_1 - i_2 - \cdots - i_{j-1} - j + 1)_{i_j}$$

These generalize the ordinary complete symmetric functions:

$$w_k(z_1, z_2, \ldots, z_n) = h_k(z_1, z_2, \ldots, z_n) + \text{lower order terms}$$

(They are special cases of the shifted Schur functions of Okounkov and Olshanski.)

A Result of Chen and Louck

Chen and Louck show that

Theorem

$$\omega_k(\mu_1, \mu_2, \ldots, \mu_l) = w_k(\mu_1, \mu_2, \ldots, \mu_l)$$

A Lemma of Verde-Star

Idea of Proof. The key result is the following

Lemma (Verde-Star, 1991)

The divided difference of the falling factorial function is

$$\frac{(x)_{m+1} - (y)_{m+1}}{x - y} = \sum_{0 \le k \le m} (x)_k (y - k - 1)_{m-k}$$

Iterating this lemma yields the result.

□

Eigenvalues Again

Theorem

The eigenvalues of the derangement graph are given by

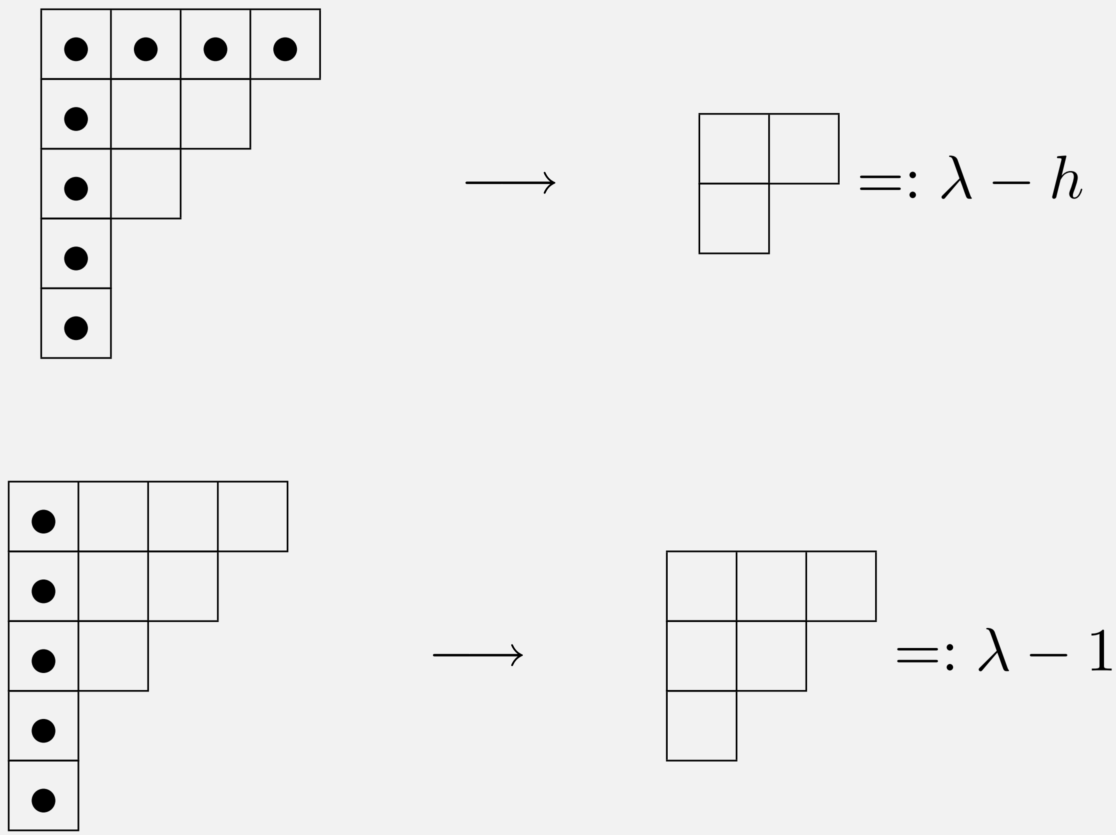
$$\eta_\lambda = \sum_{k=0}^{n} (-1)^{n-k} w_k(\mu_1, \mu_2, \ldots, \mu_l)$$

where μ *is the shifted partition associated to* λ, $n = |\lambda|$, *and* $w_k(\mu_1, \mu_2, \ldots, \mu_l)$ *is the complete factorial symmetric function defined above.*

Integrality!

Hooks and Ladders

Given a partition, say $\lambda = (4, 3, 2, 1, 1)$ we define two subpartitions:



The Main Theorem

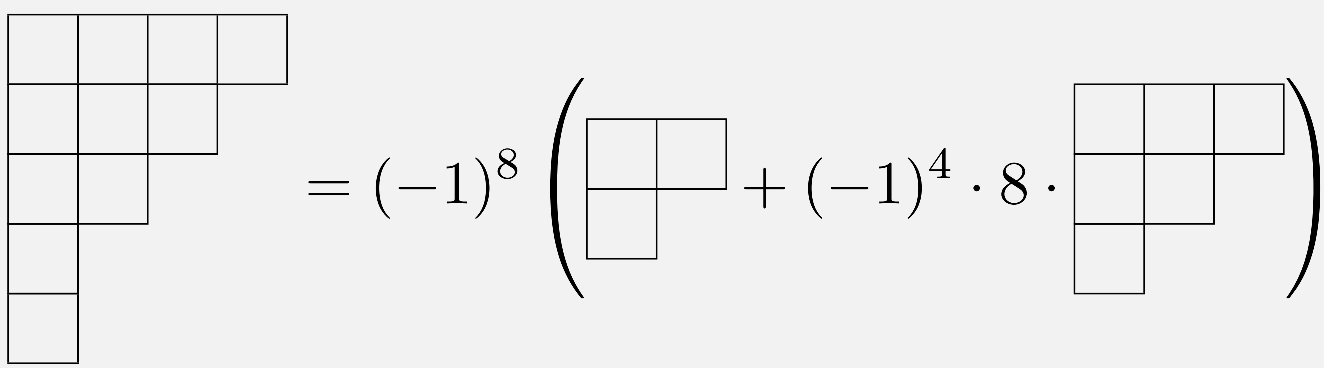
Theorem

The eigenvalues of the derangement graph satisfy the following recurrence:

$$\eta_\lambda = (-1)^h \left(\eta_{\lambda - h} + (-1)^{\lambda_1} h \eta_{\lambda - 1} \right)$$

with initial condition $\eta_\emptyset = 1$.

For example (denoting the eigenvalue by the partition):



Proof of Main Theorem

Proof. Follows from the recurrence relation for complete factorial symmetric functions (Chen and Louck):

$$
\begin{aligned}
w_k(z_1, z_2, \ldots, z_n) &= w_k(z_2 - 1, z_3 - 1, \ldots, z_n - 1) \\
&\quad + z_1 w_{k-1}(z_1 - 1, z_2 - 1, \ldots, z_n - 1)
\end{aligned}
$$

□

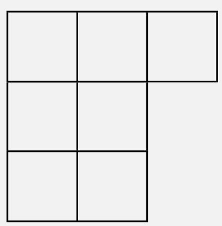
Proof of Ku's Conjecture

Theorem

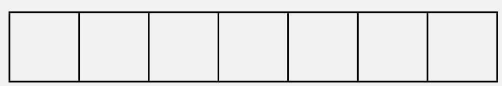
Ku's conjecture is true. Moreover, for $n \geq 5$ the least eigenvalue of the derangement graph Γ_n is uniquely achieved by the standard representation (namely, the shape $\lambda = (n-1, 1)$).

Proof (outline).

- The maximum eigenvalue is achieved by the trivial representation:



$\leq$

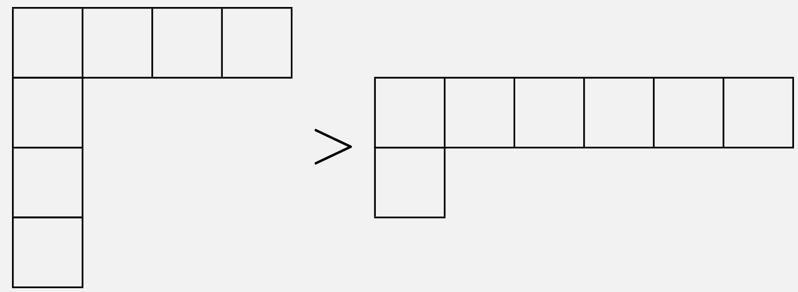


$= D_n$

General result.

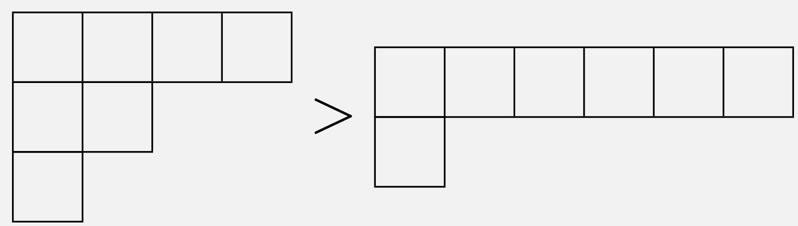
Proof continued...

- The conjecture holds for hooks.



A calculation.

- The conjecture holds for near hooks



Another calculation.

Proof continued...

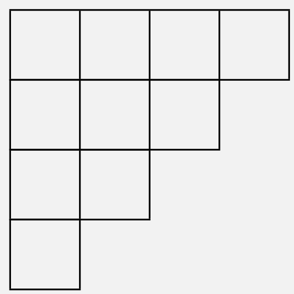
- The conjecture holds for all shapes. By above may assume λ is neither a hook $(n = h)$ nor a near hook $(n = h + 1)$. So we may assume $n \geq h + 2$ and $h > l \geq 2$. Thus...

Proof concluded

$$
\begin{aligned}
|\eta_\lambda| &= |\eta_{\lambda-h} + (-1)^{\lambda_1} h \eta_{\lambda-1}| \\
&\leq |\eta_{\lambda-h}| + h|\eta_{\lambda-1}| \\
&\leq D_{n-h} + hD_{n-l} \\
&< (1+h)D_{n-l} \\
&\leq (n-1)D_{n-l} \\
&\leq (n-1)D_{n-2} \\
&= D_{n-1} + (-1)^n \\
&\leq D_{n-1} + 1 \\
&< D_{n-1} + D_{n-2} \\
&= \frac{D_n}{n-1} \\
&= |\eta_{(n-1,1)}| \qquad \square
\end{aligned}
$$

Interesting Sequences

- Interesting sequences arise from special cases. Example: staircase shapes



$(m = 4)$

$$a_m = -\left[a_{m-2} + (-1)^m (2m-1) a_{m-1}\right]$$
$$0, -1, -5, 36, 329, -3655, \ldots$$

Sloane's online encyclopaedia of integer sequences gives this sequence modulo signs as $y(-1)$ where $y(x)$ is the so-called Bessel polynomial.

A Question

Question: Do the central characters of the symmetric group themselves obey a recurrence?