On the Chromatic Number of the Complement of a Class of Line Graphs

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ABSTRACT

Let G be a graph, $\overline{G}$ its complement, $L(G)$ its line graph, and $\chi(G)$ its chromatic number. Then we have the following

Theorem *Let G be a graph with n vertices. (i) If G is triangle free, then*

$$n - 4 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

(ii) If G is planar and every triangle bounds a disk, then

$$n - 3 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

KEYWORDS: chromatic number, line graph, planar graph, triangle-free graph, Kneser graph

1. Preliminaries

Let G be a graph, $\overline{G}$ its complement, $L(G)$ its line graph, and $\chi(G)$ its chromatic number. A *nonedge* of G is an edge of $\overline{G}$. Two nonedges of G are *adjacent* in G if they are adjacent as edges of $\overline{G}$ (*i.e.,* their endpoints intersect). They are *nonadjacent* if their endpoints are disjoint. The *clique complex* $\Delta(G)$ of G is the simplicial complex on the vertex set of G whose simplices are the cliques of G.

Following [4] we make the following definitions. For any set system $\mathcal{S}$, $KG(\mathcal{S})$ denotes the *Kneser graph* of $\mathcal{S}$, namely the graph whose vertices are the elements of $\mathcal{S}$ and whose edges are pairs of nonintersecting sets. When $\mathcal{S} = \binom{[n]}{k}$, the set of all k subsets of an n set $[n] := \{1, 2, \ldots, n\}$, we denote $KG(\mathcal{S})$ by $K_{n:k}$. $\mathrm{MIN}(\mathcal{S})$ is the system of all sets in $\mathcal{S}$ that are minimal with respect to inclusion. $\|K\|$ means the geometric realization of the simplicial complex K. $J \backslash K$ means the elements of J that are not in K.

The key result we need is Sarkaria's colouring/embedding theorem, which is a generalization of the Van Kampen-Flores theorem on the embeddability of simplices into $\mathbb{R}^d$. We recall the theorem in the form which we require:

1.1 Theorem ([4,5,6,7,8]). *Let K be a subcomplex of the $n-1$ dimensional simplex σ^{n-1}, and let $\mathcal{S} := \mathrm{MIN}(\sigma^{n-1} \backslash K)$. If*

$$d \leq n - \chi(KG(\mathcal{S})) - 2$$

then for any continuous mapping $f : \|K\| \to \mathbb{R}^d$, the images of some two disjoint faces of K intersect.

2. The Theorem

We have the following

2.1 Theorem. *Let G be a graph with n vertices. (i) If G is triangle free, then*

$$n - 4 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

(ii) If G is planar and every triangle bounds a disk, then

$$n - 3 \leq \chi\left(\overline{L(\overline{G})}\right) \leq n - 2$$

Remark. The upper bound of $n-2$ holds for any graph G, not just triangle free graphs.

Proof. A vertex of $\overline{L(\overline{G})}$ is a nonedge of G, and two vertices are adjacent in $\overline{L(\overline{G})}$ if the corresponding nonedges of G are nonadjacent in G. Let G be the empty graph on n vertices. Then $\overline{L(\overline{G})} = K_{n:2}$. By the Lovász-Kneser theorem [1,2,3,4] $\chi(K_{n:2}) = n - 2$. Adding an edge to G removes a vertex from $\overline{L(\overline{G})}$, which can only decrease its chromatic number. Hence, for any graph G, $\chi\left(\overline{L(\overline{G})}\right) \leq n - 2$.

Now let $\mathcal{S} = \mathrm{MIN}(\sigma^{n-1} \backslash G)$, where G is viewed as a one-dimensional simplicial complex. If G is triangle free, the inclusion minimal sets of $\mathcal{S}$ all have size 2, and are precisely the edges of $\overline{G}$. Hence $KG(\mathcal{S})$ is the same thing as $\overline{L(\overline{G})}$. Every graph is embeddable in $\mathbb{R}^3$, so from Theorem 1.1 we conclude that

$$n - \chi\left(\overline{L(\overline{G})}\right) - 2 < 3$$

or

$$\chi\left(\overline{L(\overline{G})}\right) \geq n - 4$$

This proves (i).

To prove the lower bound in (ii), suppose G is planar and every triangle bounds a disk. Then the simplicial complex obtained by adjoining to G all the faces bounded by triangles is homeomorphic to $\|\Delta(G)\|$. In particular, $\|\Delta(G)\|$ can be embedded in the plane. Now set $\mathcal{S} = \mathrm{MIN}(\sigma^{n-1} \backslash \Delta(G))$. The inclusion minimal nonfaces of the clique complex $\Delta(G)$ are precisely the edges of $\overline{G}$, so once again $KG(\mathcal{S})$ is just $\overline{L(\overline{G})}$. As $\|\Delta(G)\|$ embeds in the plane,

$$n - \chi\left(\overline{L(\overline{G})}\right) - 2 < 2$$

so

$$\chi\left(\overline{L(\overline{G})}\right) \geq n - 3$$

∎

3. Observations

We end with a few observations.

- The upper bound on $\chi\left(\overline{L(\overline{G})}\right)$, namely $n - 2$, is equivalent to the condition $d \geq 0$ in Theorem 1.1.
- The triangle free condition in (i) is necessary. For example, let G be $K_n - e$. Then $\overline{G}$ is a single edge, and $L(\overline{G})$ and $\overline{L(\overline{G})}$ are both a single point. As $\chi(\text{point}) = 1$, $\chi\left(\overline{L(\overline{G})}\right) < n - 4$ for any $n > 5$.

- To illustrate the theorem, let G be $K_{3,3}$, the complete bipartite graph on two sets of three vertices. Then $\overline{L(\overline{G})} = G$, and its chromatic number is $2 = 6 - 4$. Also, both bounds can be achieved in the planar case: $G = C_5$, the 5-cycle, satisfies $\overline{L(\overline{G})} = G$, and its chromatic number is $3 = 5 - 2$. On the other hand, if G is the 6-cycle plus an edge connecting two vertices a distance 3 apart on the cycle, then one can check that $\overline{L(\overline{G})}$ has chromatic number $3 = 6 - 3$.

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