It All Depends on How You Slice It: An Introduction to Hyperplane Arrangements

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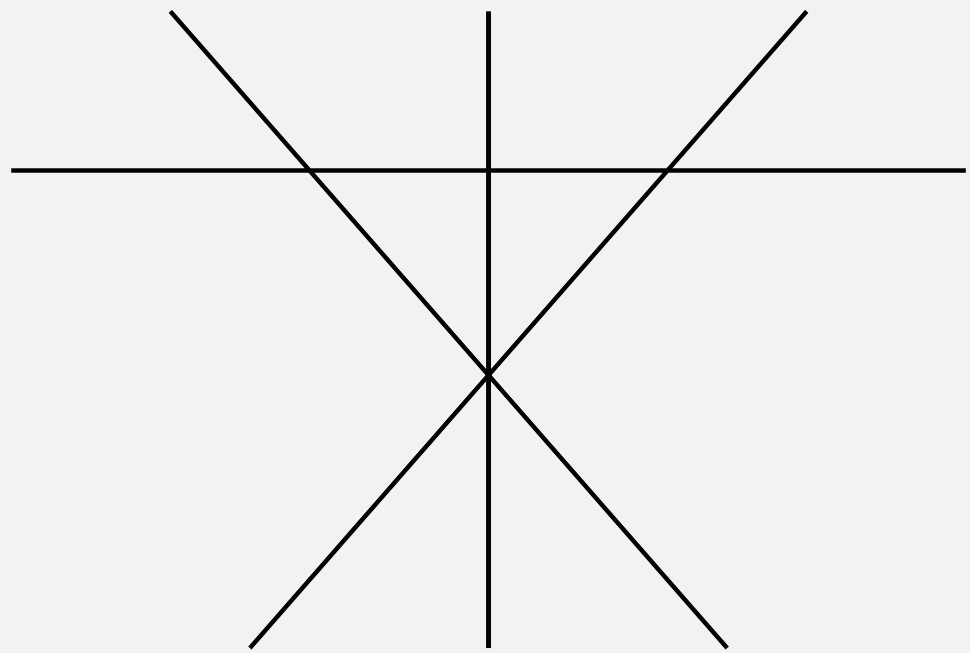
Outline

Hyperplane Arrangements

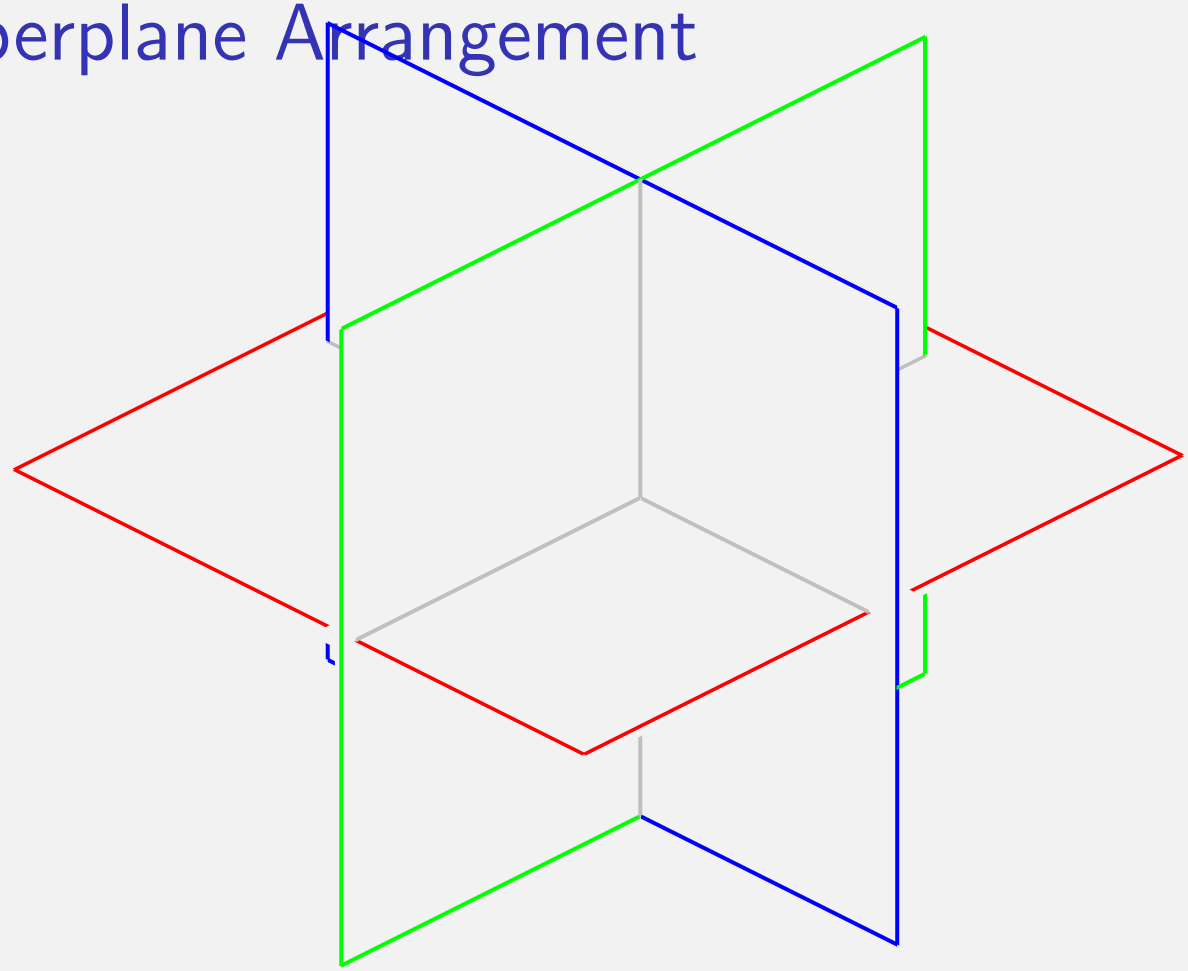
- Begin with $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.
- An affine hyperplane is the set of points in $\mathbb{R}^n$ satisfying an equation of the form $a_1x_1 + \cdots + a_nx_n = b$.
- A hyperplane arrangement is simply a collection of affine hyperplanes.

A Hyperplane Arrangement



An arrangement of four lines in the plane

Another Hyperplane Arrangement

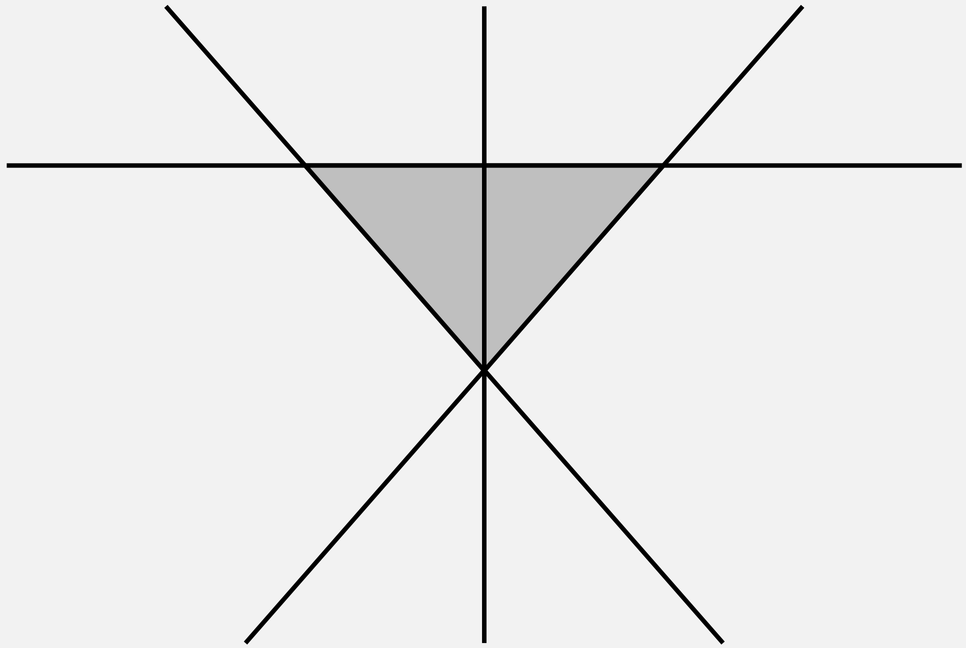


An arrangement of three planes in three space

Regions

- A region of the arrangement $\mathcal{A}$ is a connected component of the complement
$$\mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H$$
- Regions can be bounded or unbounded.
- The total number of regions is $r(\mathcal{A})$, and the number of bounded regions is $b(\mathcal{A})$.

Region Counting



$r(\mathcal{A}) = 10 \qquad b(\mathcal{A}) = 2$ (shaded)

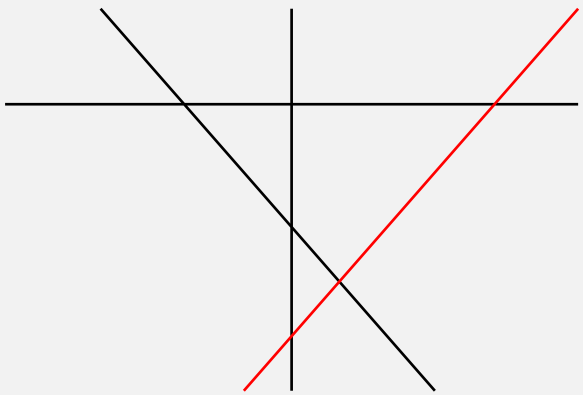
Deletion and Restriction

- Is there a better way to count regions?
- Yes!

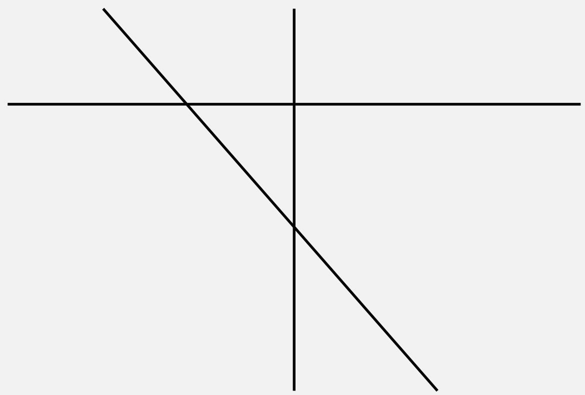
Definition

- Let $\mathcal{A}$ be an arrangement and $H \in \mathcal{A}$ a hyperplane.
- $\mathcal{A}' = \mathcal{A} \backslash H$ is called the deleted arrangement.
- $\mathcal{A}'' = \{K \cap H : K \in \mathcal{A}'\}$ is called the restricted arrangement.
- $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ is called a triple of arrangements.

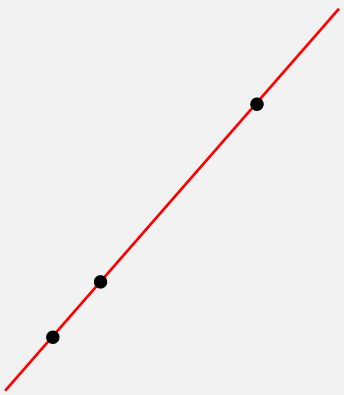
A Triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of Arrangements



$\mathcal{A}$ and H



$\mathcal{A}'$



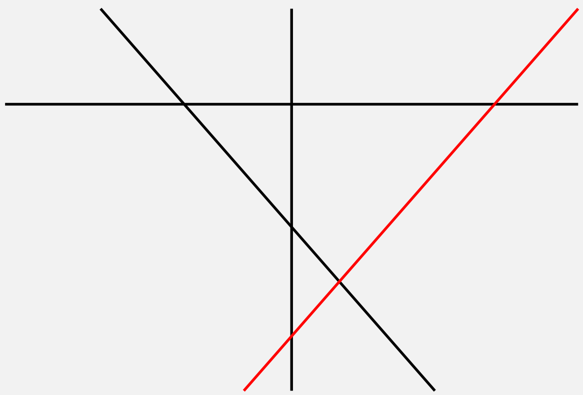
$\mathcal{A}''$

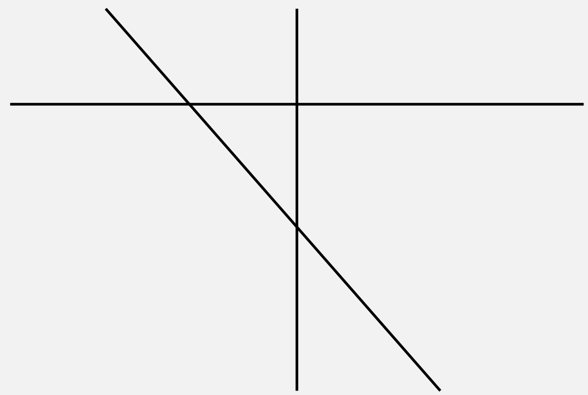
Region Counting Recurrence

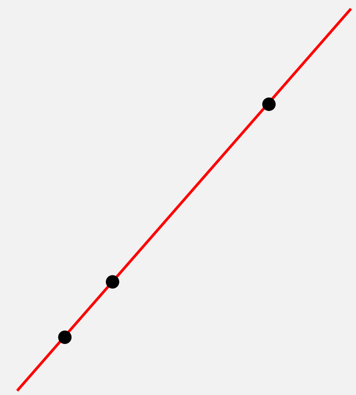
Theorem (Zaslavsky, 1975)

$$r(\mathcal{A}) = r(\mathcal{A}') + r(\mathcal{A}'')$$

Proof by Example (Don't try this at home!)







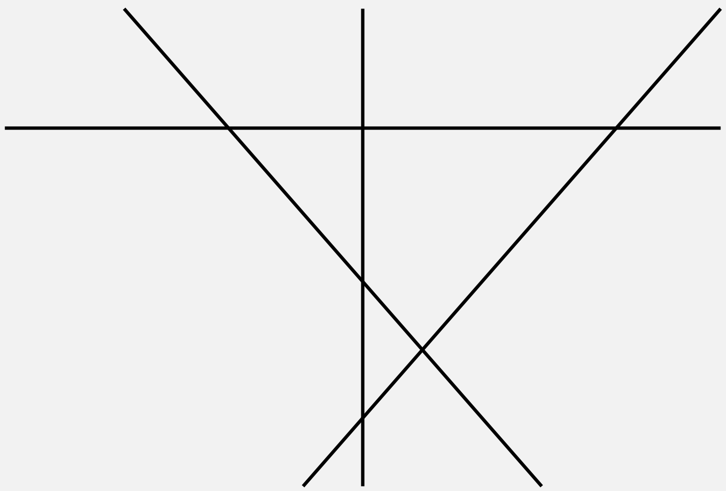
$\mathcal{A}$ and H $\qquad$ $\mathcal{A}'$ $\qquad$ $\mathcal{A}''$

$$r(\mathcal{A}) = 11 \quad = \quad r(\mathcal{A}') = 7 \quad + \quad r(\mathcal{A}'') = 4$$

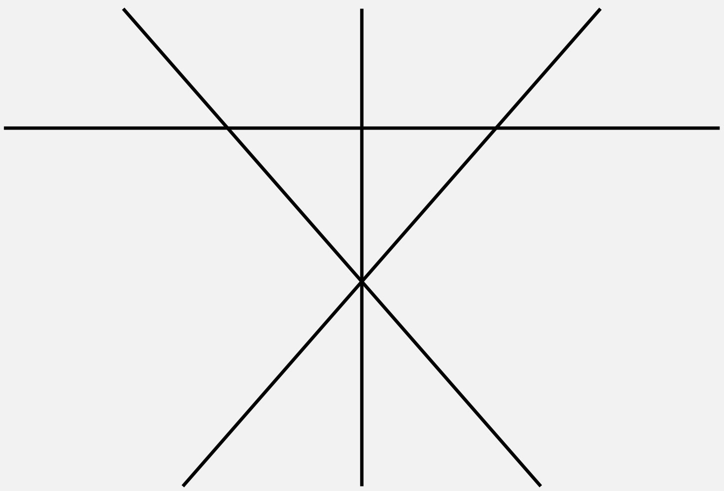
Arrangements in General Position

- Let's apply this result.
- An arrangement $\mathcal{A}$ is in general position if you can move the hyperplanes slightly and not change the number of regions.

Two Arrangements



in general position



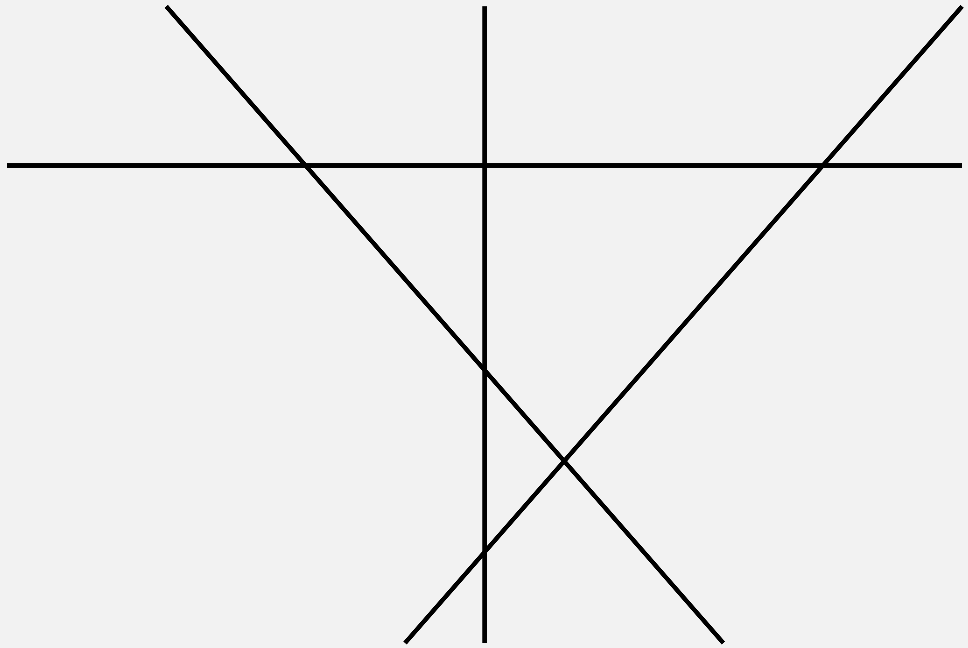
not in general position

Counting Regions of General Position Line Arrangements

- Start with an arrangement $\mathcal{A}$ of k lines in general position in the plane, and choose a particular line H.
- By hypothesis, H meets $\mathcal{A}'$ in $k-1$ points, which divide H into k regions. So $r(\mathcal{A}'') = k$.
- Hence $r(\mathcal{A}) = r(\mathcal{A}') + k$, where $\mathcal{A}'$ contains $k-1$ lines.
- By continuing to delete lines in this way, we get $r(\mathcal{A}) = r(\emptyset) + 1 + 2 + \cdots + (k-1) + k$.
- When no lines are present there is one region, so $r(\emptyset) = 1$.
- Hence, for a general position line arrangement we have

$$r(\mathcal{A}) = 1 + \sum_{i=1}^{k} i = 1 + \frac{k(k+1)}{2} = 1 + k + \binom{k}{2}.$$

Counting Regions of a General Position Line Arrangement



$$r(\mathcal{A}) = 1 + k + \binom{k}{2} = 1 + 4 + \binom{4}{2} = 11 \checkmark$$

The Number of Regions in an Arbitrary Arrangement

- We can use the recurrence (and induction) to show that

$$r(\mathcal{A}) = 1 + k + \binom{k}{2} + \binom{k}{3} + \cdots + \binom{k}{n}$$

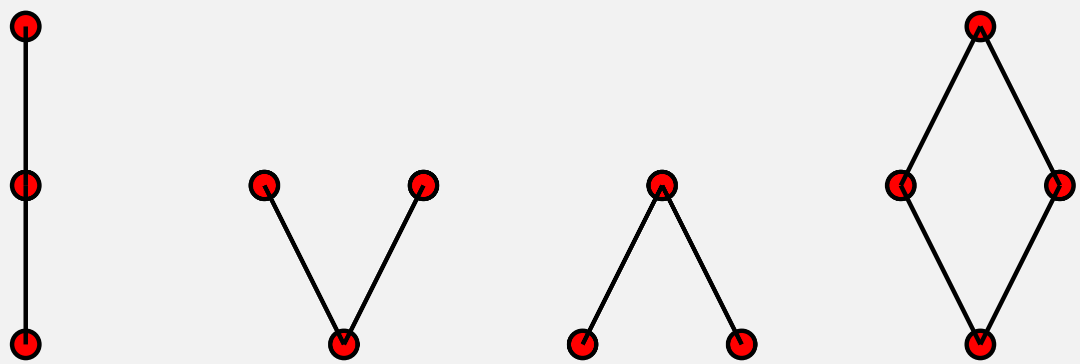
  for k hyperplanes in general position in n dimensional space. (Ludwig Schläfli, 1901)
- But what if the hyperplanes are not in general position?
- Zaslavsky developed a powerful tool to compute $r(\mathcal{A})$ in general. To describe this, we must take a long detour...

Partially Ordered Sets

A partially ordered set (poset) is a set P and a relation $\leq$ satisfying the following axioms (for all x, y, and z in P):

1. (reflexivity) $x \leq x$.
2. (antisymmetry) $x \leq y$ and $y \leq x$ implies $x = y$.
3. (transitivity) $x \leq y$ and $y \leq z$ implies $x \leq z$.
4. Posets are represented by their (Hasse) diagrams.

Hasse Diagrams

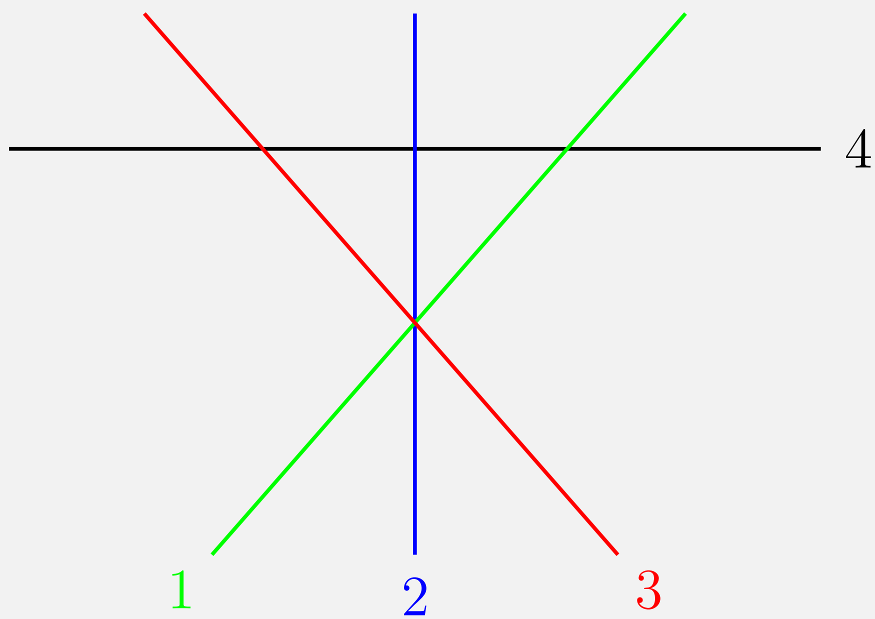


Some posets

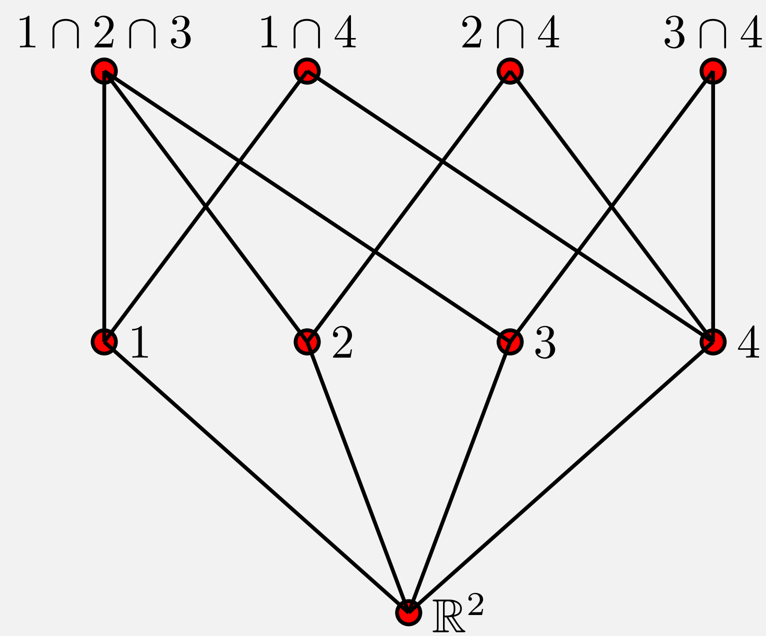
The Intersection Poset of an Arrangement

- The intersection poset $L(\mathcal{A})$ of the arrangement $\mathcal{A}$ has as its elements all the intersections of all the hyperplanes.
- It is ordered (for good reason) by reverse inclusion, so $A \subseteq B \Leftrightarrow A \geq B$.
- The minimum element is the ambient space $\mathbb{R}^n$.

An Arrangement and Its Intersection Poset



A labeled arrangement $\mathcal{A}$



Its intersection poset $L(\mathcal{A})$

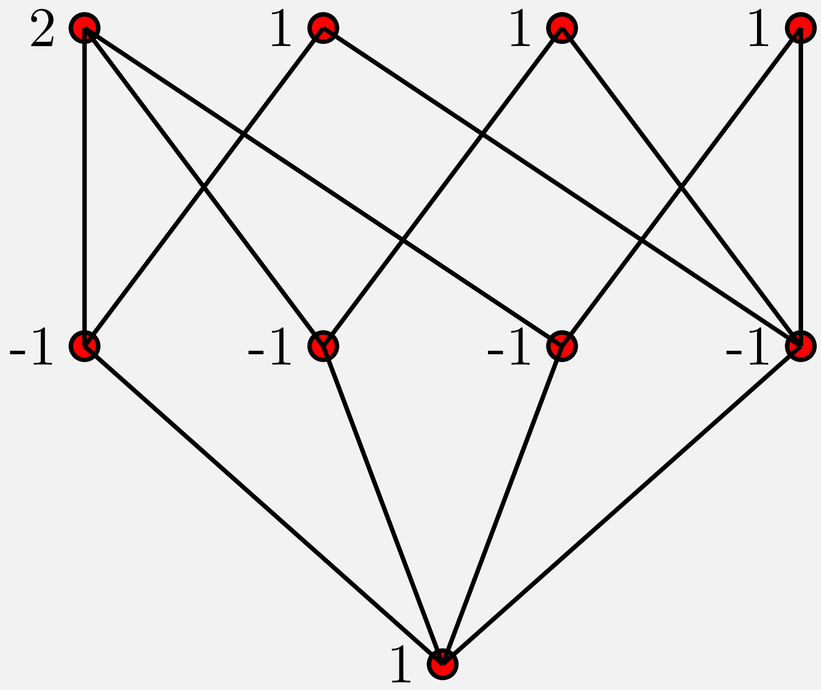
The Möbius Function for Posets

- The (closed) interval $[x, y]$ of a poset is the set of all points between x and y, including endpoints:

$$[x, y] = \{z : x \leq z \leq y\}.$$

- The Möbius function $\mu(x, y)$ is defined (recursively) on the interval $[x, y]$ by the two properties:
  1. $\mu(x, x) = 1.$
  2. $x < y \Rightarrow \sum_{z \in [x,y]} \mu(x, z) = 0$

Some Möbius Function Values



The values of the Möbius functions $\mu(\hat{0}, x)$

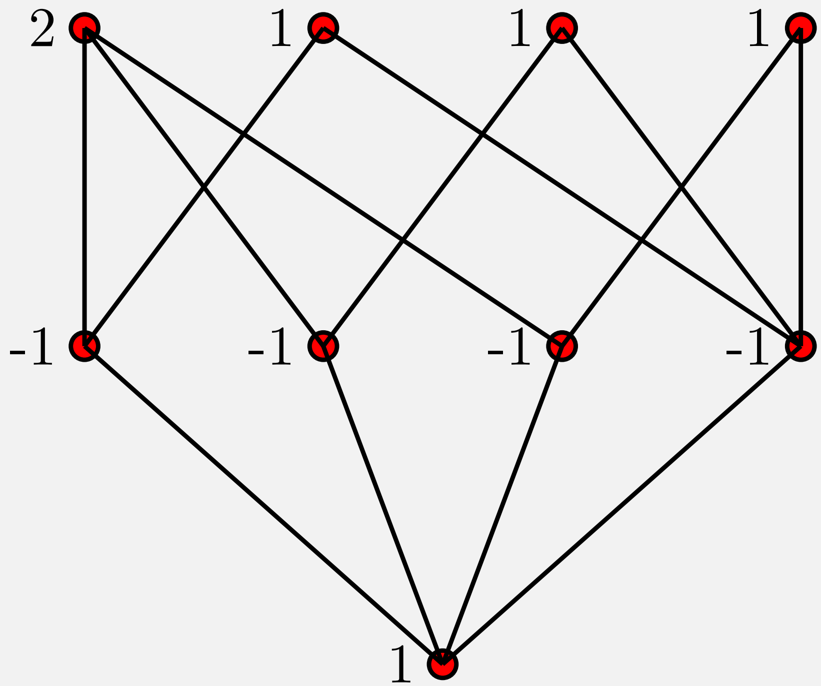
The Characteristic Polynomial

- We define the characteristic polynomial associated to the arrangement $\mathcal{A}$ by

$$\chi(\mathcal{A},q)=\sum_{x\in L(\mathcal{A})}\mu(\hat{0},x)q^{\dim(x)}$$

.

The Characteristic Polynomial of an Arrangement



$$\chi(\mathcal{A}, q) = \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) q^{\dim(x)} = q^2 - 4q + 5.$$

Zaslavsky's Theorem

Theorem (Zaslavsky, 1975)

With the definitions above

$$r(\mathcal{A}) = \sum_{x \in L(\mathcal{A})} |\mu(\hat{0}, x)| = |\chi(\mathcal{A}, -1)|$$

$$b(\mathcal{A}) = \left| \sum_{x \in L(\mathcal{A})} \mu(\hat{0}, x) \right| = |\chi(\mathcal{A}, 1)|$$

Zaslavsky's Theorem–continued

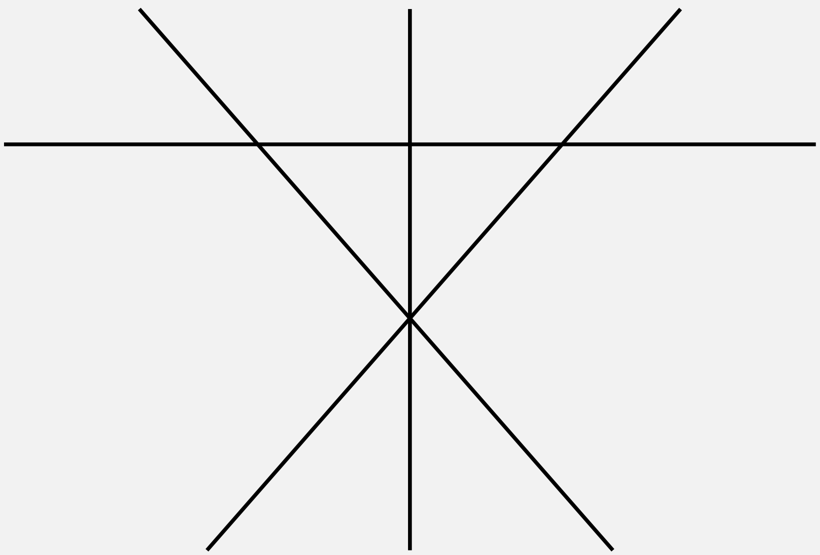
Idea of Proof.

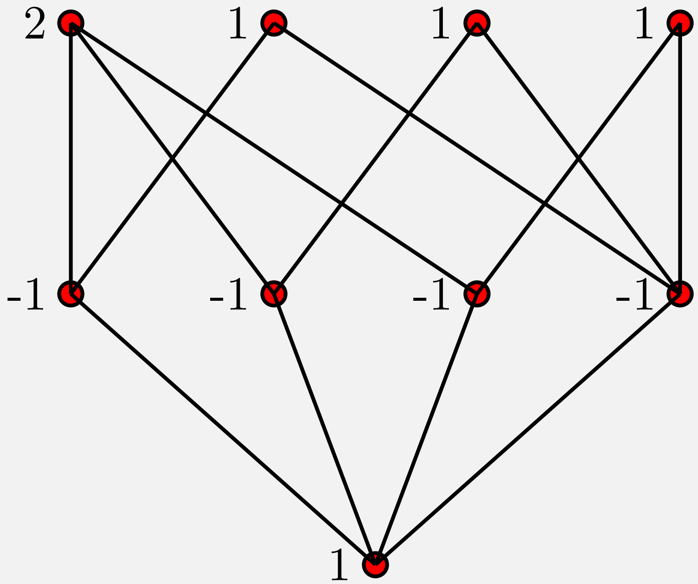
One can show that, for any triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ of arrangements,

$$\chi(\mathcal{A}, q) = \chi(\mathcal{A}', q) - \chi(\mathcal{A}'', q),$$

from which it follows that $|\chi(A, -1)|$ and $r(\mathcal{A})$ satisfy the same recurrence. As they agree on the empty set, the claim follows. A similar argument works for $b(\mathcal{A})$. □

Counting Regions *via* Zaslavsky's Theorem





$$\chi(\mathcal{A}, q) = q^2 - 4q + 5$$

$$r(\mathcal{A}) = |\chi(\mathcal{A}, -1)| = 10 \qquad b(\mathcal{A}) = |\chi(\mathcal{A}, 1)| = 2$$

Finite Field Method

- Recall that the defining equation of a hyperplane can be written $a_1x_1 + \cdots a_nx_n = b$ for some real numbers $\{a_1, a_2, \ldots, a_n, b\}$.
- In many cases of interest the numbers $a_1, a_2, \ldots, a_n, b$ are integers.
- When this holds there is a particularly nice way to compute the characteristic polynomial.
- For any positive integer q let $\mathcal{A}_q$ denote the hyperplane arrangement $\mathcal{A}$ with defining equations reduced mod q.
- Then we have the following result.

The Characteristic Polynomial for Integral Arrangements

Theorem (Crapo and Rota (1971), Orlik and Terao (1992), Athanasiadis (1996), Björner and Ekedahl (1996))

For q a sufficiently large prime

$$\chi(\mathcal{A}, q) = \#\left(\mathbb{F}_q^n - \bigcup_{H \in \mathcal{A}_q} H\right)$$

where $\mathbb{F}_q^n$ denotes the vector space of dimension n over the finite field with q elements.

Remarks

- Identifying $\mathbb{F}_q^n$ with $\{0, 1, \ldots, q-1\}^n = [0, q-1]^n$, $\chi(\mathcal{A}, q)$ is the number of points in $[0, q-1]^n$ that do *not* satisfy modulo q the defining equations of any of the hyperplanes in $\mathcal{A}$.
- We need large q to avoid lowering the rank of the defining matrix, but as both sides are polynomials in q, the two sides will agree for all q.

Reflection Arrangements

- Consider the following families of arrangements:

$$\mathcal{A}_n = \{x_i - x_j = 0 | 1 \leq i \leq j \leq n\}$$

$$\mathcal{D}_n = \mathcal{A}_n \cup \{x_i + x_j = 0 | 1 \leq i \leq j \leq n\}$$

$$\mathcal{B}_n = \mathcal{D}_n \cup \{x_i = 0 | 1 \leq i \leq n\}$$

- These are examples of reflection arrangements associated to finite Coxeter groups of types A_{n-1}, D_n, and B_n, respectively.

Computing the Characteristic Polynomial I

- What is $\chi(\mathcal{A}_n)$?
- According to the finite field method, we want the number of points in $[0, q-1]^n$ satisfying $x_i \neq x_j$ for all $1 \leq i \leq j \leq n$.
- This is the same thing as asking for vectors $(x_1, x_2, \ldots, x_n)$ all of whose entries are distinct mod q.
- Well, we can pick x_1 in q ways, then x_2 in $q-1$ ways, and so on. Thus

$$\chi(\mathcal{A}_n, q) = q(q-1)(q-2)\cdots(q-n+1).$$

- It follows that $\boxed{r(\mathcal{A}_n) = n!}$ (and $b(\mathcal{A}_n) = 0$).

Computing the Characteristic Polynomial II

- What is $\chi(\mathcal{B}_n)$?
- Now we want to count the points satisfying $x_i \neq x_j$, $x_i \neq -x_j$, and $x_i \neq 0$.
- Since we do not allow 0, there are only $q-1$ (nonzero) choices for the first entry, $q-3$ nonzero choices for the second entry (because we must avoid the first entry and its negative), *etc.*.
- Thus

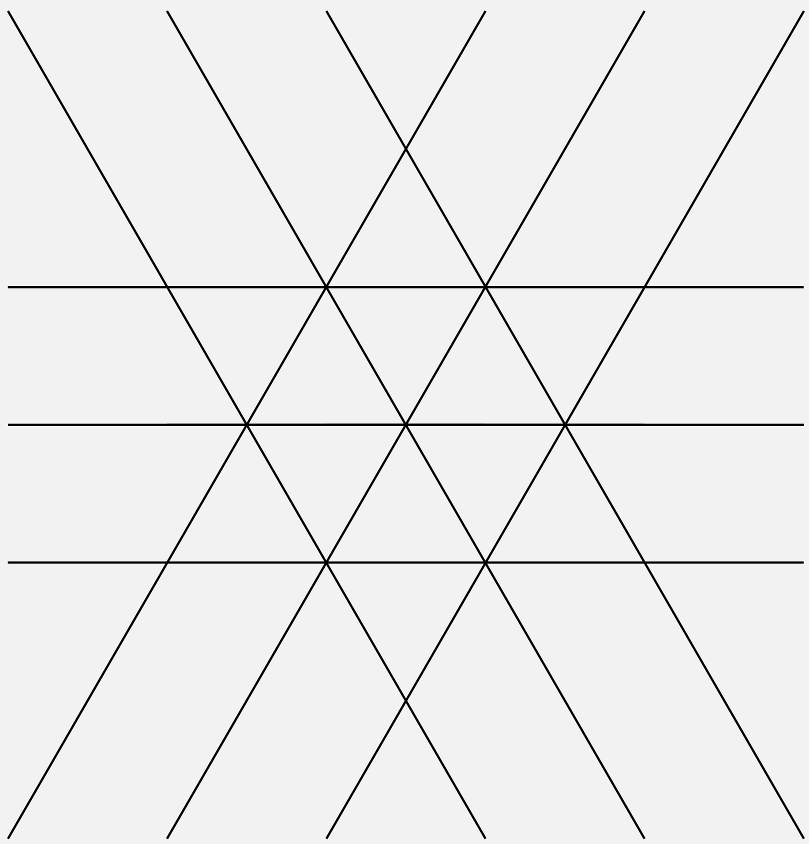
$$\chi(\mathcal{B}_n, q) = (q-1)(q-3)\cdots(q-2n+1).$$

- It follows that $\boxed{r(\mathcal{B}_n) = 2^n n!}$ (and $b(\mathcal{B}_n) = 0$).

The Catalan Arrangement

$$\mathcal{C}_n = \{x_i - x_j = -1, 0, +1 | 1 \leq i \leq j \leq n\}$$

C_3 (projected)



The Catalan Arrangement

- Using the finite field method you can show that

$$\chi(\mathcal{C}_n, q) = q(q-n-1)(q-n-2)\cdots(q-2n+1)$$

- Hence

$$r(\mathcal{C}_n) = n!C_n \qquad \text{and} \qquad b(\mathcal{C}_n) = n!C_{n-1}$$

  where

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

  is the n^{th} Catalan number.
- As of April 1, 2008, Richard Stanley has listed 164 combinatorial interpretations of C_n on his website.

Fundamental Open Question

When does the characteristic polynomial factor completely over the integers?